1. As reigning champion of *The Ultimate Extreme Roller Coaster Happy Fun Hour* you have recently been hired to help design rides in amusement park. One of the rides that you are working on involves a large bowl-like surface on which people will be strewn into. In particular, the surface is \( z = \frac{1}{2}(x^2 + y^2) \) for \( 0 \leq z \leq 2 \). You are working to ensure the supports in place will be sufficient to support the weight. In particular the design calls for the density to be increase with the height, i.e., \( \delta = z \). Help estimate the total mass of the surface by finding \( \iint \delta \, d(SA) \).

\[
\iint_{S} \delta \, d(SA) = \int_{R} \frac{1}{2}(x^2+y^2) \sqrt{(x)^2 + (y)^2 + 1} \, dA
\]

Switch to polar coordinates:

\[
= \int_{0}^{2\pi} \int_{0}^{2} \frac{1}{2} r^2 \sqrt{r^2+1} \, r \, dr \, d\theta
\]

Note: \( \int_{0}^{2\pi} \, d\theta = 2\pi \)

\[
= \pi \int_{0}^{2} r^2 \sqrt{r^2+1} \, r \, dr
\]

\[
= \pi \int_{1}^{5} (u-1) \sqrt{u} \, \frac{1}{2} \, du
\]

\[
= \frac{\pi}{2} \int_{1}^{5} (u^{3/2} - u^{1/2}) \, du
\]

\[
= \frac{105\pi}{3} + \frac{2\pi}{15}
\]
2. Let $C$ be the curve $(\cos(\pi t^2) + t^{137}, e^{t(1-t)} - \sin(\pi t/2))$ for $0 \leq t \leq 1$, and similarly let $F = (3y^2 + \cos(x + y), -\cos(x + y))$. Find

\[
\int_C F \cdot n \, ds = \int_C \langle M, N \rangle \cdot \left( \frac{dy}{dx} - \frac{dx}{dy} \right)
\]

(It might be useful to recall that $n = \left( \frac{dx}{ds}, -\frac{dy}{ds} \right)$.)

$C(0) = (0, 0) = (1, 1)$

$C(1) = (-1 + 1, 1 - 1) = (0, 0)$

\[
\int_C -N \, dx + M \, dy
\]

\[
= \int_C \left( \frac{\cos(x+y)}{2y} \right) dx + \left( \frac{3y^2 + \cos(x+y)}{2y} \right) dy
\]

\[
\rightarrow f = \sin(x+y) + C(y)
\]

\[
\frac{df}{dy} = \cos(x+y) + C'(y) = 3y^2 + \cos(x+y)
\]

\[
\rightarrow C'(y) = 3y^2 \rightarrow C(y) = \frac{3}{3}y^3
\]

\[
\rightarrow f = \sin(x+y) + y^3
\]

So, $\int_C F \cdot n \, ds = f(0,0) - f(1,1) = 0 - (\sin(2) + 1)$

\[
= -1 - \sin 2
\]
3. Let \( C \) be the curve consisting of straight line segments that visits the following points in the indicated order:

\[(0,0) \to (2,1) \to (4,0) \to (5,2) \to (4,4) \to (2,3) \to (0,4) \to (-1,2) \to (0,0).\]

Find

\[
\oint_C \left( y^2 + \cos(x^3) - xy \right) \, dx + \left( 2xy + \frac{1}{1 + e^{2y}} \right) \, dy.
\]

(Hint: draw the region; use Green's Theorem then reinterpret the resulting integral as trying to find a parameter of the region with density \( \delta = 1 \); compute this parameter in a different way.)

\[
\iint_S (2y) - (2y - x) \, dA
\]

\[
\iint_S x \, dA
\]

If \( S = 1 \):

\[
\bar{X} = \frac{\iint_S x \, dA}{\text{Area}} \quad \implies \quad \iint_S x \, dA = 2 \cdot 16 = 32
\]

2 by symmetry

16, pieces rearrange to form 4x4 square
4. Let $S$ be the solid consisting of the set of points satisfying $1 \leq x^2 + y^2 \leq 4$ and $0 \leq z \leq 3$. Find

$$
\iiint_S \left( e^y - 2xz, z^4 - 2y - \sin(e^x), z^2 + 3z - \ln(1 + x^2) \right) \cdot n \, d(SA)
$$

$$
\iiint_S (2z - z^4 + 2z + 3) \, dV
$$

Shape is cylinder with smaller cylinder missing

$$
volume = \frac{\pi \cdot 2^2 \cdot 3 - \pi \cdot 1^2 \cdot 3}{12\pi} = \frac{9\pi}{3\pi} = \frac{9\pi}{3\pi} = \frac{9\pi}{3\pi}
$$

$$
\frac{1}{12}\pi
$$
5. Let $G$ be the surface of the hyperboloid of one sheet $1 + z^2 = x^2 + y^2$ with $-1 \leq z \leq 2$ and let the normal vectors $n$ point out away from the $z$-axis. Find

$$
\int_G \langle yz, 3x + z^2 - y \frac{\partial}{\partial z} \cos(x^2) + \sin(y^2) \rangle \cdot T \, ds.
$$

\[ \text{change surface to more convenient} \]

\[ \text{circle of radius } \sqrt{5} \text{ at } z = 2 \]
\[ \text{normal } \langle 0, 0, 1 \rangle \]

\[ \text{circle of radius } \sqrt{2} \text{ at } z = -1 \]
\[ \text{normal } \langle 0, 0, 1 \rangle \]

\[ \nabla \times \langle yz, 3x + z^2 - y, \cos(x^2) + \sin(y^2) \rangle \]

\[ = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3x + z^2 - y & \cos(x^2) + \sin(y^2) \end{vmatrix} \]

\[ = \langle 2yz, y^2 - 2z, y + 2x \sin x^2, 3 - 2 \rangle \]

So

$$
\int_G \langle \cdot \cdot \cdot \rangle \cdot T \, ds = \int_{H_1} \oint \langle \cdot \cdot \cdot \rangle \cdot \langle 0, 0, 1 \rangle \, d(sA) + \int_{H_2} \oint \langle \cdot \cdot \cdot \rangle \cdot \langle 0, 0, -1 \rangle \, d(sA) \]

$$

\[ = \int_{H_1} \int_{3z = 4 \text{ on } H_1} 4 \, d(sA) + \int_{H_2} \int_{3z = 1 \text{ on } H_2} (-1) \, d(sA) \]

since $z = -1$

since $z = 2$

\[ = 4 (\text{Area of } H_1) - (\text{Area of } H_2) = 4 \cdot 2\pi - \frac{5\pi}{3} \]

\[ = \boxed{\frac{3\pi}{3}} \]