This test is closed book and closed notes. No sophisticated calculator is allowed for this test. For full credit show all of your work (legibly!). Each problem is worth 10 points (a total of 50 points).

1. Given the following:

\[
\begin{align*}
\int_1^4 \int_{2y}^{\sqrt[3]{y}} f(x, y) \, dx \, dy &= 2 \\
\int_0^2 \int_{x^2}^{\sqrt{8x}} f(x, y) \, dy \, dx &= 9 \\
\int_0^4 \int_0^{y^{2/8}} f(x, y) \, dx \, dy &= 4
\end{align*}
\]

There are exactly two regions \( R_1 \) and \( R_2 \) besides the three given above for which \( \iint_{R_1} f(x, y) \, dA \) and \( \iint_{R_2} f(x, y) \, dA \) can be determined. For the smaller of the two regions set up an iterated integral and determine the value.

\[
\begin{align*}
\text{Region I:} & \quad x = \sqrt[3]{y} \\
\text{Region II:} & \quad x = \frac{4-y}{3} \\
\text{Region III:} & \quad y = x^2
\end{align*}
\]

Can find this piece \( \text{II} + \text{III} - \text{I} \)

\[
\int_0^1 \int_{x^2}^{4-3x} f(x, y) \, dy \, dx = 11
\]

Can determine integral over larger region

1. 
2. 
3. 
4. 
5. 
\( \Sigma \)
2. Find \( \int_0^1 \int_{x^{2/3}}^{\sqrt[3]{y}} 8x \cos(y^4) \, dy \, dx \).

\[
\begin{align*}
  &= \int_0^1 \left[ \int_0^{\sqrt[3]{y}} 8x \cos(y^4) \, dx \right] \, dy \\
  &\quad= \int_0^1 4x^2 \cos(y^4) \bigg|_{x=0}^{x=\sqrt[3]{y}} \, dy \\
  &\quad= \int_0^1 4y^{3/2} \cos(y^4) \, dy \\
  &\quad= \sin(y^4) \bigg|_{y=0}^{y=1} \\
  &\quad= \sin(1)
\end{align*}
\]
3. Consider the solid region which consists of the points inside a sphere of radius 1 centered at the origin and satisfying $z \geq 0$. Given the density $\delta(x, y, z) = z^2$ for this solid, determine the center of mass of this solid. (Use symmetry where possible; it might be useful to recall in cylindrical coordinates that $z = z$ and in spherical coordinates $z = \rho \cos \phi$.)

$$M = \iiint_S (\text{density}) \, dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho \cos \phi)^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{\pi/2} \cos^2 \phi \, d\phi \right) \left( \int_0^1 \rho^4 \, d\rho \right) = \frac{2\pi}{15}$$

$$M_{xy} = \iiint_S (\text{distance})(\text{density}) \, dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho \cos \phi) (\rho \cos \phi)^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{\pi/2} \cos^3 \phi \, d\phi \right) \left( \int_0^1 \rho^5 \, d\rho \right) = \frac{\pi}{12}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\frac{\pi}{12}}{\frac{2\pi}{15}} = \frac{5}{8}$$

Center of mass = \((0, 0, \frac{5}{8})\)
4. Before setting out on their quest to reclaim the gold, Thorin Oakenshield needed to start making arrangements to transport all of the gold back to his home in the Blue Mountains (Dwarves are known for their optimism in accomplishing impossible tasks). To get an estimate for the number of pack animals needed to transport the treasure he needs to do some quick computations on the amount of treasure for which the volume gives an accurate measure (treasure has a consistent density). Given that Smaug’s lair where the treasure is kept can be described as a semicircular room with a radius of 10 meters, the highest point on the treasure is at the midpoint of the straight wall forming one side of the semicircle and the height at distance \( r \) from this point is \( \frac{10}{1 + r^2} \) meters, then find the total volume.

\[
\text{Finding volume over semicircle}
\]  
\[ 
\iint_R (\text{height}) \, dA 
\]  
\[ 
= \int_0^{\pi} \int_0^{10} \frac{10}{1 + r^2} \, r \, dr \, d\theta 
\]  
\[ 
= \int_0^{\pi} 5 \ln(1 + r^2) \bigg|_{r=0}^{r=10} \, d\theta 
\]  
\[ 
= \int_0^{\pi} 5 \ln(101) \, d\theta 
\]  
\[ 
= 5 \ln(101) \theta \bigg|_{\theta=0}^{\theta=\pi} 
\]  
\[ 
= 5 \pi \ln(101) 
\]
5. Rewrite (but do NOT evaluate) the following integral

\[ \int_0^{\sqrt{1/2}} \int_{y^2}^{1-y^2} \frac{8xy + 8y^3}{1 + (x - y^2)^3} \, dx \, dy \]

by using the change of variables \( u = x - y^2 \) and \( v = x + y^2 \).
(For reference the region for the integral is shown on the right.)

\[ u = x - y^2 \quad \quad u + v = 2x \quad \quad x = \frac{u + v}{2} \]
\[ v = x + y^2 \quad \quad v - u = 2y^2 \quad \quad y = \sqrt{\frac{v - u}{2}} \]

Function:

\[ \frac{8xy + 8y^3}{1 + (x - y^2)^3} = \frac{8y(x + y^2)}{1 + (x - y^2)^3} = \frac{8\left(\frac{v - u}{2}\right)^{1/2}}{1 + u^3} \cdot v \]

Bounds:

\[ x = y^2 \quad \quad x - y^2 = 0 \quad \quad u = 0 \]
\[ x = 1 - y^2 \quad \quad x + y^2 = 1 \quad \quad v = 1 \]
\[ y = 0 \quad \quad \sqrt{\frac{v - u}{2}} = 0 \quad \quad v = 0 \]

Jacobian:

\[ \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{4}(\frac{v - u}{2})^{-1/2} \\ \frac{1}{2} & \frac{1}{4}(\frac{v - u}{2})^{1/2} \end{vmatrix} = \frac{1}{4}(\frac{v - u}{2})^{-1/2} + \frac{1}{4}(\frac{v - u}{2})^{1/2} \]

\[ \int_0^{\sqrt{1/2}} \int_{y^2}^{1-y^2} \frac{8xy + 8y^3}{1 + (x - y^2)^3} \, dx \, dy = \int_0^1 \int_0^1 \frac{\sqrt{v}}{1 + u^3} \cdot \frac{1}{\sqrt{4\pi}} \, du \, dv \]

= \int_0^1 \int_0^1 \frac{2v}{1 + u^3} \, du \, dv
Super Duper Extra Special Bonus Problem for the Midterm Regrade (+5 points)

Let us go back and revisit the center of mass for the sphere. Suppose that the density for the sphere of radius 1 centered at the origin and satisfying \( z \geq 0 \) has density \( \delta(x, y, z) = z^\alpha \) where \( \alpha > -1 \). When \( \alpha \) is close to \(-1\) the density is heavily amassed at the bottom and \( \bar{z} \approx 0 \) as \( \alpha \) gets close to \( \infty \) the density at the bottom vanishes and starts to accumulate at the top and \( \bar{z} \approx 1 \).

Determine the unique \( \alpha > -1 \) so that \( \bar{z} = \frac{1}{2} \).

\[
\iiint z^\beta \, dV = \int_0^{2\pi} \int_0^\pi \int_0^1 \left( \rho \cos \phi \right)^\beta \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} \frac{\pi}{2} (\cos \phi)^\beta \sin \phi \, d\theta \left[ \frac{1}{\beta+1} \left. \frac{u^{\beta+1}}{\beta+1} \right|_0^1 - \frac{1}{\beta+3} \right] = \frac{2\pi}{(\beta+1)(\beta+3)}
\]

\[
\frac{1}{\bar{z}} = \frac{\iint z \cdot \bar{z}^\alpha \, dV}{\iint \bar{z}^\alpha \, dV} = \frac{\frac{2\pi}{(\alpha+2)(\alpha+4)}}{\frac{2\pi}{(\alpha+1)(\alpha+3)}} = \frac{(\alpha+1)(\alpha+3)}{(\alpha+2)(\alpha+4)}
\]

\[
(\alpha+2)(\alpha+4) = 2(\alpha+1)(\alpha+3)
\]

\[
\alpha^2 + 6\alpha + 8 = 2\alpha^2 + 8\alpha + 6
\]

\[
0 = \alpha^2 + 2\alpha - 2
\]

\[
\alpha' = \frac{-2 \pm \sqrt{4-4(-2)}}{2} = \frac{-2 \pm \sqrt{12}}{2} = -1 \pm \sqrt{3}
\]

\[
\alpha' = \frac{-1 - \sqrt{3}}{-1 + \sqrt{3}} \text{ (so no good)}
\]

\[
\alpha = \sqrt[3]{3} - 1
\]