Integration works component by component (make sure to get all constants)

\[ \int \langle f(t), g(t), h(t) \rangle \, dt = \langle \int f(t) \, dt, \int g(t) \, dt, \int h(t) \, dt \rangle \]

\( \int (\text{velocity}) \, dt = \text{position/dispacement} \)

\( \int (\text{speed}) \, dt = \text{distance} \)

\[ \int_a^b \| r'(t) \| \, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt \]

* Look for squares inside \( \sqrt{\phantom{0}} \).
* "Massage" to integrate.
**GOAL**

Improve our understanding of motion

Osculating plane

\[ T(t) = \frac{r'(t)}{\|r'(t)\|} \quad \text{unit vector pointing in direction of motion} \]

\[ N(t) = \frac{T'(t)}{\|T'(t)\|} \quad \text{“unit normal vector”} \]

\[ B(t) = T(t) \times N(t) \quad \text{“binormal unit vector”} \]
\[ T(t) = \frac{r'(t)}{\|r'(t)\|} \quad \text{OR} \quad v(t) = \|r'(t)\| T(t) \]

\[ \alpha(t) = \frac{d}{dt}(v(t)) = \frac{d}{dt}(\|r'(t)\| T(t)) \]

\[ = \frac{d}{dt}(\|r'(t)\| T(t)) + \|r'(t)\| \frac{d}{dt}(T(t)) \]

\[ = \frac{d}{dt}(\|r'(t)\| T(t)) + \|r'(t)\| T'(t) \]

\[ \quad \text{points in direction of motion, i.e.,} \quad T(t) \]

\[ \quad \text{points in direction perpendicular to motion, i.e.,} \quad N(t) \]

\[ = \alpha_T(t) T(t) + \alpha_N(t) N(t) \]

**Scalar valued functions indicating how much of acceleration is going in these directions.**
\[ a_T = \| r'' \| \cos \theta = \frac{\| r'' \| \| r' \| \cos \theta}{\| r' \|} \]

\[ a_N = \| r'' \| \sin \theta = \frac{\| r'' \| \| r' \| \sin \theta}{\| r' \|} \]
Putting this together

\[ a = a_T T + a_N N \]

- \( a_T \): Component of acceleration in the direction of motion
- \( a_N \): Component of acceleration perpendicular to the direction of motion

\[ a_T = \frac{r' \cdot r''}{|| r'' ||} T \]
\[ a_N = \frac{|| r' \times r'' ||}{|| r'' ||} N \]
Example: Given \( r(t) = <\cos t, \sin t, t> \), find \( a_T \) and \( a_N \).

\[
r'(t) = <-\sin t, \cos t, 1>
\]
\[
r''(t) = <-\cos t, -\sin t, 0>
\]
\[
||r'|| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}
\]
\[
r' \cdot r'' = \sin t \cos t - \cos t \sin t + 0 = 0
\]
\[
[r' \times r''] = \begin{vmatrix} i & j & k \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = <\sin t, -\cos t, 1>
\]
\[
||r' \times r''|| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}
\]
\[
a_T = \frac{r' \cdot r''}{||r'||} = \frac{0}{\sqrt{2}} = 0
\]
\[
a_N = \frac{||r' \times r''||}{||r'||} = \frac{\sqrt{2}}{\sqrt{2}} = 1
\]
Curvature ("bendiness")

Goal is to measure how quickly the curve is turning.

Note, we are interested in the curve, not how we move along the curve.

low curvature  high curvature

Idea: look at what is happening to T.
\[ K(t) = \left\| \frac{dT}{ds} \right\| = \text{How fast the unit tangent vector changes along the curve} \]

Pro: good intuitive explanation

Con: Hard to compute this way!!!
Simplification 1 - Chain rule

\[ \left\| \frac{dT}{ds} \right\| = \left\| \frac{dT}{dt} \cdot \frac{dt}{ds} \right\| \]

\[ = \left\| \frac{dT}{dt} \right\| \left| \frac{dt}{ds} \right| \]

\[ = \frac{\left\| \frac{dT}{dt} \right\|}{\left| \frac{ds}{dt} \right|} \]

\[ = \frac{\left\| T' \right\|}{\left\| r' \right\|} \]

\[ \text{change in distance} \]

\[ \frac{\text{change in distance}}{\text{change in time}} \]

\[ = \left\| r'(t) \right\| \]
Simplification 2 - Find $|\mathbf{T'}|$

1. $N(t) = \frac{T'(t)}{|T'(t)|}$

2. $a(t) = \frac{d}{dt}(v(t)) = \frac{d}{dt}(\|r'(t)\| T(t))$

3. $a(t) = \left(\frac{r' \cdot r''}{\|r''\|}\right) T(t) + \left(\frac{\|r' \times r''\|}{\|r''\|^2}\right) N(t)$

\[\|r''\| \|\mathbf{T'}\| = \frac{\|r' \times r''\|}{\|r''\|} \implies \|\mathbf{T'}\| = \frac{\|r' \times r''\|}{\|r''\|^2}\]

$K(t) = \|\frac{d}{ds}T\| = \frac{\|\mathbf{T'}\|}{\|r'\|} = \frac{\|r' \times r''\|}{\|r''\|^3} \leftarrow 3$!!
Example

Find $K(t)$ for $r(t) = \langle t + t^2, t - t^2, 2t \rangle$.

\[ r' = \langle 1 + 2t, 1 - 2t, 2 \rangle \]
\[ r'' = \langle 2, -2, 0 \rangle \]
\[ r' \times r'' = \begin{vmatrix} i & j & k \\ 1 + 2t & 1 - 2t & 2 \\ 2 & -2 & 0 \end{vmatrix} \]
\[ = \langle 4, 4, -4 \rangle \]

\[ K = \frac{11 \langle 4, 4, -4 \rangle}{11 \langle 1 + 2t, 1 - 2t, 2 \rangle} \]
\[ = \frac{\sqrt{16 + 16 + 16}}{(1 + 4t + 4t^2 + 1 - 4t + 4t^2 + 4)} \]
\[ = \frac{4 \sqrt{3}}{(1 + 6 + 8t^2)} \]