The final reviewing!!

From earlier in the course-
Arc length of $C = (x(t), y(t), z(t))$ for $a \leq t \leq b$

$$\int_{a}^{b} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt$$

Surface area of $z = f(x,y)$ over $R$

$$\iint \sqrt{(f_x)^2 + (f_y)^2 + 1} \, dA$$
We can generalize these to line integrals and surface integrals.

Basic idea: Break into parts, find contribution of each part, put it altogether.

\[ \int_C f(x,y,t) \, ds = \int_a^b f(x(t),y(t),z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt \]

**“ds”**

reexpress everything in terms of single variable \( t \).

\[ \iint_G g(x,y,t) \, d(SA) = \iint_R g(x,y,f(x,y)) \sqrt{(fx)^2 + (fy)^2 + 1} \, dA \]

**“d(SA)”**

reexpress everything in terms of variables \( x,y \).

Applications: Mass, Center of mass, Moments, etc...
\[ z = f(x, y) \]

\[ C = (x + L \psi(x)) \]

\[ \int_C f(x, y) \, ds \]
Vector valued functions

\[ F(x,y,z) = M(x,y,z) \hat{i} + N(x,y,z) \hat{j} + P(x,y,z) \hat{k} \]

\[ = Mi + Nj + PK \]

\[ = (M, N, P) \]

Del operator: \( \nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \)

Gradient: \( F = \nabla f \) \quad \{ f = \text{potential} \}

\( F = \text{conservative} \)

Divergence: \( \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \)

Curl: \( \nabla \times F = \left| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{array} \right| \)}
\[ \int_C M\,dx + N\,dy + P\,dz = \int_C \mathbf{F} \cdot \mathbf{r} \, ds \]

- to evaluate express everything in terms of \( t \)
- useful for work

- the "\( \sqrt{\cdot} \)" terms in \( T \) and in \( ds \) cancel. Woohoo!!

**Big Idea #1**

Line integrals of conservative functions are easy.

\[ \int_C \nabla \mathbf{F} \cdot \mathbf{r} \, dr = f(b) - f(a) \]

\[ \text{BYOC} \]

- independent of curve \( C \)
How to recognize —

* Look for crazy curve / function

\[ \int_C M \, dx + N \, dy \]
\[ \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]  
(Similar in 3d)

Reconstructing \( f \)

\[ \int_C M \, dx + N \, dy + P \, dz \]

\[ f = \int M \, dx + C(y,z) \]

Now use \( N \) to find \( C(y,z) = \ldots + D(z) \)

Finally use \( P \) to find \( D(z) \)

Don't forget to get end points of curve and evaluate \( f(\text{end}) - f(\text{start}) \)
This is a simple example of
\[ \int_{\Omega} \omega = \oint_{\partial \Omega} \omega \]

integral on boundary of function

\[ \int_{R} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA \]

integral in interior of some derivative

**Big Idea # Z - Green**

Orientation: To the Left
Look for closed curve in the plane. Check orientation, if orientation is backward this changes sign.

Works for regions with holes.

Good rule of thumb—
If something looks ridiculous, it probably is. Use the ideas.
Special case:
\[ \oint \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{R}} (\text{div} \mathbf{F}) \, dA \]

**BIG IDEA #3 - GAUSS**

**DIVERGENCE THEOREM**

\[ \iiint_{V} \mathbf{F} \cdot d\mathbf{S} = \iiint_{T} (\text{div} \mathbf{F}) \, dV \]

**Key word:** **Solid**

After changing to \( T \) use techniques of integration, i.e., cylindrical/spherical, etc...
Special case:
\[ \oint_{C} F \cdot Tds = \iint_{R} (\text{curl}F) \cdot K \, dA \]

Big Idea #4 - Stokes

Orientation: To the LEFT

What to look for: Surface with boundary

BYOS
Example Let $C$ be the curve corresponding to the intersection of $x^2 + y^2 = 1$ and $z = y^2 - x^2$ oriented counterclockwise when viewed from above. Find

$$\oint_C \left< -y, \arctan z, \frac{y}{1+z^2} \right> \cdot T \, ds$$

$$= \iint_G (\nabla \times \mathbf{F}) \cdot n \, dS$$

Where $G$ is the unit circle.

\[
\begin{bmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{2}} \\
-\arctan z & \frac{y}{1+z^2} & -y \arctan z
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{1+z^2} & 0 & 0 \\
\frac{1}{1+z^2} & 0 & 0 \\
-\frac{1}{1+z^2} & 0 & 0
\end{bmatrix}
\]

\[
\left< \left( \frac{1}{1+z^2} \right) i + 0 j + 0 k \right> \rightarrow \left< 0, 0, 1 \right>
\]
$$\iint_{\{0,0,1\}} \mathbf{d}(SA) \cdot n$$

$$G$$

\[
\iint_{\mathbb{R}} 1 \, dA = \pi
\]
Recall

Surface $G \uparrow \varepsilon = f(x, y)$ over $R$

\[ \iint_{G} \mathbf{M} \cdot \mathbf{n} = \iiint_{R} (-Mf_x - Nf_y + P) \, dA \]

replace any $\varepsilon$ by $f(x, y)$

this is any vector field, so perfectly fine to have this be $\nabla \times \mathbf{F}$