FINAL on
THURSDAY, DEC. 17
7-9 pm

Good: extra time to study
Bad: can't leave until Dec. 18
From last time...

\[ \int_c^d \int_a^b f(x,y) \, dx \, dy = \int_a^b \int_c^d f(x,y) \, dy \, dx \]

Region \( R \)

\[ a \leq x \leq b \]
\[ c \leq y \leq d \]
If our region is complicated we can break it into several pieces.

Example  Let $R$ be the region $y \leq x^2$ and $0 \leq x \leq 2 - y^2$. Rewrite $\iint_R f(x,y) \, dA$ as a sum of iterated integrals.

\[
\begin{align*}
&\int_{-\sqrt{2}}^{0} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f(x,y) \, dx \, dy \\
&\quad + \int_{\sqrt{2}}^{2} \int_{0}^{\sqrt{2-y^2}} f(x,y) \, dx \, dy
\end{align*}
\]
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Example: Let $R$ be the region $y \leq x^2$ and $0 \leq x \leq 2-y^2$. Rewrite $\int \int_R f(x,y) \, dA$ as a sum of iterated integrals.

\[
\int_0^2 \int_{y^2}^{\sqrt{2-x}} f(x,y) \, dy \, dx
\]

\[
\int_{y^2}^{\sqrt{2-x}} \int_0^x f(x,y) \, dx \, dy
\]

"dy \, dx"
Example: Find \[
\iint_R e^{y^2} \, dy \, dx
\]

Idea: \[
\int_0^4 \int_0^{\frac{x}{2}} e^{y^2} \, dy \, dx
\]

\[
R \begin{cases} 0 \leq x \leq 4 \\ \frac{x}{2} \leq y \leq 2 \end{cases}
\]

\[
\int_0^2 \int_{\frac{x}{2}}^2 e^{y^2} \, dy \, dx = \int_0^2 \left( e^{y^2} \bigg|_0^x \right) \, dy
\]

\[
= \int_0^2 2ye^{y^2} \, dy
\]

\[
= \int_0^4 e^u \, du = e^u \bigg|_{u=0}^{u=y^2} = e^4 - 1
\]
The previous problem gives us a new tool:

**Changing bounds**

\[
\int_a^b \int_{\psi_1(x)}^{\psi_2(x)} f(x,y) \, dy \, dx = \int_c^d \int_{\phi_1(y)}^{\phi_2(y)} f(x,y) \, dx \, dy
\]

1. Write down current bounds
2. Draw a picture indicating the region (and how we are "slicing")
3. Label all boundary curves both as \( x = \phi(y) \) and \( y = \psi(x) \)
4. Use the picture to determine how to break into parts and on each part what the bounds are. Work from the outside in.
Example

Rewrite the following

\[ \int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} f(x,y) \, dx \, dy = \int_{-1}^{0} \int_{x^2-1}^{0} f(x,y) \, dy \, dx \]

\[-\sqrt{y+1} \leq x \leq \sqrt{y+1} \]

\[-1 \leq y \leq 0 \]

\[ x = \sqrt{y+1} \]
\[ x^2 = y+1 \]
\[ y = x^2 - 1 \]

\[ y = 0 \]

\[ y = x^2 - 1 \]
Surface area

Given $z = f(x, y)$, what is the total surface area over $R$?

Basic plan
1. Break into pieces
2. Approximate each piece
3. Add together all of the pieces
0. Break into pieces (small!)

When we zoom in on the piece as it corresponds to the surface we will see the following:

- Nearly flat!
- Can use tangent planes to approximate.
Surface area of parallelogram

Surface area of part of our surface

Recall

\[ \text{Area} = \| \mathbf{u} \times \mathbf{v} \| \]
\[ \langle 0, ay, f_y ay \rangle \sim \langle \Delta x, 0, f_x \Delta x \rangle \]

\[ \langle \Delta x, 0, \Delta x f_x \rangle \times \langle 0, ay, ay f_y \rangle \]

\[
\begin{array}{ccc}
i & j & k \\
\Delta x & 0 & \Delta x f_x \\
0 & ay & ay f_y & 0 & ay
\end{array}
\]

\[
= \langle -f_x \Delta x ay, -f_y \Delta x ay, \Delta x ay \rangle
\]

\[ = \langle -f_x, -f_y, 1 \rangle \Delta x ay \rightarrow \Delta A \]
Surface area on a small piece

\[ \approx \| \langle -f_x, -f_y, 1 \rangle \| \, \Delta A \]

\[ = \sqrt{(f_x)^2 + (f_y)^2 + 1} \, \Delta A \]

So total surface area

\[ \approx \sum \sqrt{(f_x)^2 + (f_y)^2 + 1} \, \Delta A \]

\[ \longrightarrow \int_R \int \sqrt{f_x^2 + f_y^2 + 1} \, dA \]

Surface area of \( z = f(x, y) \)
over the region \( R \).
Example

Set up, but do not evaluate, an integral for the surface area of

\[ z = x^3 y - 3xy^2 = f(x,y) \]

where \[ R = [0,3] \times [-1,4] \]

\[ \iint\limits_R \sqrt{f_x^2 + f_y^2 + 1} \, dA \]

\[ = \int_{-1}^{1} \int_{0}^{3} \sqrt{\left(3x^2 y - 3y^2\right)^2 + \left(x^3 - 6xy^2\right)^2 + 1} \, dx \, dy \]