From last time...

Properties of the gradient:
1. $\nabla f$ points in direction of greatest increase
2. $\|\nabla f\|$ = rate of greatest increase
3. $-\nabla f$ points in direction of greatest decrease
4. $-\|\nabla f\|$ = rate of greatest decrease
5. $\nabla f \perp$ level curves / surfaces

Tangent plane methods:
1. For $z = f(x,y)$ at $(a,b)$
   
   $z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)

2. For $F(x,y,z) = c$ at $(a,b,c)$
   
   $\nabla F(a,b,c) \cdot (x-a, y-b, z-c) = 0

Linear approximation (for $z = f(x,y)@ (a,b)$)

$\Delta z \approx \frac{\partial f}{\partial x}(a,b) \Delta x + \frac{\partial f}{\partial y}(a,b) \Delta y$
We can also estimate tolerances.

Example: Your company is a leader in widget manufacturing (widget's are sphere balls). You have a contract to deliver widgets of radius 3 inches. No widget machine is perfect, there are always some small defects. Given that the widgets that are sold must be within 1 in$^3$ of the correct volume, estimate the tolerance for the radius (i.e., how much can we let or be and still make a good widget).
\[ V = \frac{4}{3} \pi r^3 \]
\[ \Delta V \approx 4\pi r^2 \Delta r \]
\[ \ell \approx 4\pi^2 \Delta r \]
\[ \Delta r \approx \frac{1}{4\pi^2} \approx 36, 3.14 \text{ m} \]
\[ \approx 0.01 \]
Can we get a better approximation?

\[ f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + \text{ERROR} \]

need to put our effort here.

Calculus II Flashback - Taylor series polynomial

Near \( x = a \)

\[ f(x) = f(a) + \text{ERROR} \]

Only matching function

\[ f(x) = f(a) + f'(a)(x-a) + \text{ERROR} \]

Matching function + derivative aka tangent line

\[ f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \text{ERROR} \]

Matching function, first derivative, and second derivative
Good - match function + first derivatives
(linear approximation)

Better - match function, first derivatives, and second derivatives

Near \((a,b)\) we have

\[
 f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) \\
+ \frac{\partial^2 f}{\partial x^2}(a,b)\frac{(x-a)^2}{2} + \frac{\partial^2 f}{\partial x\partial y}(a,b)(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(a,b)\frac{(y-b)^2}{2}
\]

Second derivatives @ \((a,b)\)

\[
\text{Second order Taylor polynomial at } (a,b)
\]

ERROR
Example: Find the second order Taylor polynomial at \((0,0)\) for \(f(x,y) = e^x \sin y + e^{-y} \cos x\)

\[
\begin{align*}
    f(0,0) & = 1 \\
    f_x(0,0) & = 0 \\
    f_y(0,0) & = 0 \\
    f_{xx}(0,0) & = -1 \\
    f_{yy}(0,0) & = 1 \\
    f_{xy}(0,0) & = 1
\end{align*}
\]

\[
f(x,y) \approx 1 + 0 \cdot x + 0 \cdot y + (-1) \frac{x^2}{2} + (1)xy + (1) \frac{y^2}{2}
\]

\[
\approx 1 - \frac{x^2}{2} + xy + \frac{y^2}{2}
\]
**TAKING IT TO THE MAX!!**

An important application of calculus is optimization, i.e., finding min's and max's.

\[ f(x,y) \text{ has a local max at } (a,b) \text{ if } f(a,b) \geq f(x,y) \text{ for all } (x,y) \text{ near } (a,b). \]

\[ f(x,y) \text{ has a local min at } (a,b) \text{ if } f(a,b) \leq f(x,y) \text{ for all } (x,y) \text{ near } (a,b). \]

\[ f(x,y) \text{ has a global max at } (a,b) \text{ if } f(a,b) \geq f(x,y) \text{ for all } (x,y). \]

\[ f(x,y) \text{ has a global min at } (a,b) \text{ if } f(a,b) \leq f(x,y) \text{ for all } (x,y). \]
Finding location of min's/max's

Recall $\nabla f$ points in direction of greatest increase and $-\nabla f$ points in direction of greatest decrease.

If we are at a min or max then one of the following is true.

$\Rightarrow ① \nabla f = 0 \quad \leftarrow$

$② \nabla f$ is undefined

$③$ on the boundary.

$\leftarrow$ critical points
Example: Find all critical points of 

\[ f(x, y) = 3x^3 + y^2 - 9x + 4y + 137 \]

\[ \nabla f = \mathbf{0} \quad \iff \quad f_x = 0 \quad \text{and} \quad f_y = 0 \]

\[ f_x = 9x^2 - 9 = 0 \quad \Rightarrow \quad x = \pm 1 \]

\[ f_y = 2y + 4 = 0 \quad \Rightarrow \quad y = -2 \]

\[ (1, -2) \quad (-1, -2) \]
Being a critical point $\Rightarrow$ max, min

$z = x^2 - y^2$ at $(0,0)$

Critical point: SADDLE

New goal: Classification

Calculus I approaches

* First derivative test
* Second derivative test
At a critical point \( f_x = f_y = 0 \),

so Taylor polynomial becomes

\[
f(x,y) \approx f(a,b) + f_x(a,b) \frac{(x-a)^2}{2} + f_y(a,b)(y-b) + \frac{f_{yy}(y-b)^2}{2}
\]

Let \( x-a = \Delta x \), \( y-b = \Delta y \)

\[
\approx f(a,b) + \frac{1}{2} \left( f_{xx}(\Delta x)^2 + 2f_{xy} \Delta x \Delta y + f_{yy}(\Delta y)^2 \right)
\]

\[
\approx f(a,b) + \frac{1}{2} \begin{bmatrix} \Delta x & \Delta y \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}
\]

\[
\text{matrix multiplication}
\]
Awesomeness from matrices

Given matrix \( \begin{bmatrix} A & B \\ B & C \end{bmatrix} \) there are numbers \( \lambda_1, \lambda_2 \) (eigenvalues) with

\[
\begin{bmatrix} \Delta x & \Delta y \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \lambda_1 (\Delta x + \beta \Delta y)^2 + \lambda_2 (\gamma \Delta x + \delta \Delta y)^2
\]

\[
\frac{\lambda_1 \Delta x + \beta \Delta y}{\gamma \Delta x + \delta \Delta y} = \frac{\lambda_2 \Delta x + \beta \Delta y}{\gamma \Delta x + \delta \Delta y}
\]

If \( \lambda_1, \lambda_2 > 0 \), the minimum is achieved.

If \( \lambda_1, \lambda_2 < 0 \), the maximum is achieved.

If one \( \lambda \geq 0 \) and one \( \lambda < 0 \), it is a saddle point.

If at least one \( \lambda \leq 0 \), it is a doh!
In particular we only need the sign of these special numbers.

Last bit of awesome:

$$\lambda_1, \lambda_2 = AC - B^2$$

Back to our problem:

$$f(x, y) = f(a, b) + \frac{1}{2} \begin{bmatrix} ax & ay \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} ax \\ ay \end{bmatrix}$$

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

Discriminant, sign is important!
$D < 0 \Rightarrow \lambda_1, \lambda_2$ have different signs, so at a saddle.

$D > 0 \Rightarrow \lambda_1, \lambda_2$ have same signs, either both positive or both negative.

MIN

MAX

To determine min or max, we look in "slice" and use second derivative test:

$F_{xx} > 0 \Rightarrow$ min in slice $\Rightarrow$ min

$F_{xx} < 0 \Rightarrow$ max in slice $\Rightarrow$ max

$D = 0 \Rightarrow$ not enough information!
(take more math)
Second partials test

If $\nabla f(a,b) = 0$ (i.e. a critical point)
compute $D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$

1. $D(a,b) < 0 \Rightarrow (a,b)$ a saddle

2. $D(a,b) > 0$
   \[ \begin{cases} 
   f_{xx} > 0 \Rightarrow (a,b) \text{ a min} \\
   f_{yy} > 0 \\
   f_{xx} < 0 \Rightarrow (a,b) \text{ a max} \\
   f_{yy} < 0 
   \end{cases} \]

3. $D(a,b) = 0 \Rightarrow \text{we don't know!}$
1. Find critical points
   \[ \nabla f = 0 \leftarrow \text{system of equations} \]
   (substitute, factor)

2. Classify critical points
   second partials test

Example: Find and classify critical points of \( f(x,y) = x^2 - 3xy + 2y^2 + x - y + 17 \).

\[
\begin{align*}
    f_x &= 2x - 3y + 1 = 0 \quad (x^3) \\
    f_y &= -3x + 4y - 1 = 0 \quad (x^2)
\end{align*}
\]

\[
\begin{align*}
    -3x + 4y - 1 &= 0 \\
    2x - 3y + 1 &= 0 \\
    2x &= 2 \quad \text{or} \quad x = 1 \\
    \text{C.P.} \quad (1,1)
\end{align*}
\]

\[
D = f_{xx} f_{yy} - (f_{xy})^2 = 2 \cdot 4 - (-3)^2 = -1
\]

SADDLE
Example: Find and classify critical points for the function

\[ g(x,y) = x^4 - 4xy - 7x^2 + 4y^2 + 4x - 8y + 20 \]

\[ g_x = 4x^3 - 4y - 14x + 4 = 0 \]
\[ g_y = -4x + 8y - 8 = 0 \]

\[ 2x^3 - 2y - 7x + 2 = 0 \]

\[ 2x^3 - (x+2) - 7x + 2 = 0 \]
\[ 2x^3 - 8x = 0 \]
\[ 2x(x^2 - 4) = 0 \]

\[ g_{xx} = (12x^2 - 14) \]
\[ g_{yy} = 0 \]
\[ g_{xy} = -4 \]

\[ D = 8(12x^2 - 14) - 64 \]

Critical points: (0, 1) \( \text{MIN} \), (2, 2) \( \text{MIN} \), (-2, 0) \( \text{SADDLE} \)