From last time...

Limits tell us what should happen at some point based on what is happening nearby.

\[ \lim_{(x,y) \to (a,b)} f(x,y) = L \]

Strategies:

1. Is \( f(x,y) \) continuous? If so plug in the point. If \( \neq L \) then done.

2. Do we get different answers depending on our approach? Then DNE.

3. Look for a way to rewrite and cancel, or bound (i.e. use squeezing)
Idea — look at cross sections

$z = f(x, y)$

Fix $x$ and look at cross section

$g(y) = f(x, y)$

Now a single variable function so we can take derivative!
If we hold $x$ fixed, then our function depends only on the variable $y$, i.e., $g(y) = f(x, y)$

$$g'(y) = \lim_{h \to 0} \frac{g(y+h) - g(y)}{h}$$

$$= \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

$$= \frac{\partial f}{\partial y} (x, y)$$

This is known as the partial derivative of $f$ with respect to $y$. 
\[ \frac{\partial f}{\partial y}(x, y) = \lim_{{h \to 0}} \frac{f(x, y+h) - f(x, y)}{h} \]

\[ \text{"partial f partial y"} \]

\[ \text{how is the output changing as we change } y \text{ (but hold } x \text{ fixed)} \]

\[ \frac{\partial f}{\partial x}(x, y) = \lim_{{h \to 0}} \frac{f(x+h, y) - f(x, y)}{h} \]

\[ \text{"partial f partial x"} \]

\[ \text{how is the output changing as we change } x \text{ (but hold } y \text{ fixed)} \]
Notation

\( \partial \) = partial, indicating a derivative of a function with more than one input

\( d \) = derivative of a function with exactly one input

\[ \frac{df}{dx}(x) = f'(x) \]

\[ \frac{\partial F}{\partial x}(x,y) \]

\[ f'(xy) \] (crossed out) \[ F_x(x,y) \]
Basic rule: When taking a partial derivative with respect to some variable, treat all other variables as constants.

Example:

\[ f(x,y) = x^2 + x \arctan y + ye^{137 \sin(y^2 + 7)} \]

What is \( \frac{\partial f}{\partial x} \)?

\[ \frac{\partial f}{\partial x} = 2x + \arctan y + 0 \]
Example: Find all of the first order partial derivatives for \( g(x,y,z) = 6x^4 + y \tan x + x y^2 \).

\[
\frac{\partial g}{\partial x} = 24x^3 + y \cdot \frac{1}{1 + x^2} + y^2 x \cdot (y^2 - 1)
\]

\[
\frac{\partial g}{\partial y} = \arctan x + (\ln x) \cdot y^2 + z y^2 - 1
\]

\[
\frac{\partial g}{\partial z} = (\ln x) x y^2 + \ln(y) y z^2
\]
Example Find all of the first order partial derivatives for 
\[ g(x, y, z) = 12x^3 y z + e^{\arctan x} + (x)^{\sin z} \]

\[ \frac{\partial g}{\partial x} = 36x^2 y z + \frac{e^{\arctan x}}{1 + x^2} + (\sin z)^x \cdot x^{\sin z - 1} \]

\[ \frac{\partial g}{\partial y} = 12x^3 z \]

\[ \frac{\partial g}{\partial z} = 12x^3 y + \ln(x) x^{\sin z} \cdot \cos z \]

\[ x^{\sin t} = (e^{\ln x})^{\sin t} = e^{(\ln x \cdot \sin z)} \]
Higher order partial derivatives

\[ \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = f_{xx} \]

\[ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = f_{yx} \]

\[ \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = f_{xy} \]

\[ \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = f_{yy} \]

*NOT ALWAYS EQUAL!*  
*But in our class they are*
Example Find all second order partial derivatives of

\[ f(x,y) = x^2 e^{xy} + y \cos x \]

\[
\frac{\partial f}{\partial x} = 2xe^{xy} + x^2 ye^{xy} - y \sin x
\]

\[
\frac{\partial f}{\partial y} = x^3 e^{xy} + \cos x
\]

\[
\frac{\partial^2 f}{\partial x^2} = 2e^{xy} + 2xye^{xy} + 2xye^{xy} + x^2 y^2 e^{xy} - yc\cos x
\]

\[
\frac{\partial^2 f}{\partial x \partial y} = 3x^2 e^{xy} + x^3 ye^{xy} - \sin x
\]

\[
\frac{\partial^2 f}{\partial y \partial x} = 2x^2 e^{xy} + x^2 e^{xy} + x^3 ye^{xy} - x^2 ye^{xy}
\]

\[
\frac{\partial^2 f}{\partial y^2} = xe^{xy}
\]

\[
\frac{\partial^2 f}{\partial x \partial y} = xe^{xy}
\]
Of course we can take higher order partial derivatives

\[ \frac{\partial^9 g}{\partial x^4 \partial y^3 \partial z^2} (x, y, z) \]
Partial differential equations

expressions connecting how things are changing

\[ u = u(x,t) \]
\[ u_t u_{xx} = u_x u_{tx} \]

Example: Verify \( u(x,t) = \sin(x + \sin t) \) is a solution.

\[ u_t = \cos(x + \sin t) \cdot \cos t \]
\[ u_{xx} = -\sin(x + \sin t) \]
\[ u_x = \cos(x + \sin t) \]
\[ u_{tx} = \cos t \cdot (-\sin(x + \sin t)) \]

SOLN! Woohoo!
Differentiable

= can take a derivative
= locally looks flat
$f(x,y) = 0$

$z = \sqrt{|x|} \sqrt{|y|}$

Not enough for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ to exist.

$\frac{\partial z}{\partial x} (0,0) = 0$

$\frac{\partial z}{\partial y} (0,0) = 0$

At $(0,0)$ this does not look flat.