This test is closed book and closed notes. A (graphing) calculator is allowed for this test but cannot also be a communication device (i.e., your cellphones or tablets are not calculators). Answer each question completely using exact values unless otherwise indicated. Show your work (legibly); answers without work and/or justifications will not receive credit. Place your final answer in the provided box. Each problem is worth 10 points for a total of 70 points.
1. Change the order of integration and then find the exact value of

\[
\int_0^1 \int_y^{y^{1/3}} 4 \sin(2x^2 - x^4) \, dx \, dy.
\]

\[
x = y^{1/3} \quad \text{or} \quad y = x^3
\]

\[
\int_0^1 \int_{x^3} y \sin(2x^2 - x^4) \, dy \, dx = \int_0^1 \int_0^x y \sin(2x^2 - x^4) \, dy \, dx
\]

\[
= \int_0^1 4 \sin(2x^2 - x^4) \left. y \right|_{y=x^3}^{y=x} \, dx
\]

\[
= \int_0^1 4 (x - x^3) \sin(2x^2 - x^4) \, dx
\]

\[
= \int_0^1 \frac{4 (x-x^3) \sin(2u)}{4x^2 - 4x^3} \, du
\]

\[
= \int_0^1 \sin(u) \, du = -\cos(u) \bigg|_{u=0}^{u=1}
\]

\[
= -\cos(1) - (-\cos(0)) = 1 - \cos(1)
\]

Answer: \[
\int_0^1 \int_{x^3} x \sin(2x^2 - x^4) \, dy \, dx = 1 - \cos(1)
\]
2. Find the unique point, \((x, y, z)\), on the surface \(x^2 - 2xy + 3y^2 - z = 20\) where the tangent plane is parallel to the lines

\[
\begin{align*}
x &= t - 2 \\
y &= t + 3 \\
z &= -4t + 5
\end{align*}
\quad and \quad
\begin{align*}
x &= 3t - 4 \\
y &= 2t - 1.
\end{align*}
\]

(Hint: what relationship will the gradient vector have with the lines?)

\[
\nabla g = \langle 2x - 2y, -2x + 6y, -1 \rangle
\]

\[
\vec{0} = \langle 2x - 2y, -2x + 6y, -1 \rangle \circ \langle 1, 1, -4 \rangle \quad \text{direction of the lines}
\]

\[
\vec{0} = \langle 2x - 2y, -2x + 6y, -1 \rangle \circ \langle 3, 2, 2 \rangle
\]

\[
\begin{align*}
0 &= (2x - 2y) + 3(-2x + 6y) + 2 = 4y + 4 \\
\Rightarrow \quad y &= -1
\end{align*}
\]

\[
\begin{align*}
0 &= 3(2x - 2y) + 2(-2x + 6y) - 2 = 2x + 6y - 2 \\
\Rightarrow \quad 2x - 8 &= 0 \\
\Rightarrow \quad x &= 4
\end{align*}
\]

\[
\begin{align*}
y^2 - 2 \cdot 4 \cdot (-1) + 3 \cdot (-1)^2 - z &= 20 \\
16 + 8 + 3 - 2 &= 20 \\
27 - z &= 20 \\
z &= 7
\end{align*}
\]

Answer: \((4, -1, 7)\)
3. Find the center of mass for the region of points with \(1 \leq x^2 + y^2 \leq 4\) and \(y \geq 0\) given that \(\delta(x, y) = y\). (Recall: \(\cos^2 \theta = \frac{1}{2} (1 + \cos(2\theta))\) and \(\sin^2 \theta = \frac{1}{2} (1 - \cos(2\theta))\)).

\[
\bar{y} = \frac{M_x}{M} = \frac{\iint y \text{ (density)} \, dA}{\iint \text{ (density)} \, dA}
\]

\[
M = \iint y \, dA \\
= \int_0^\pi \int_0^2 r \sin \theta \, d\theta \, dr \\
= \left[ -\cos \theta \bigg|_{\theta = 0}^{\theta = \pi} \right] \left( \frac{1}{3} r^3 \bigg|_{r = 1}^{r = 2} \right) \\
= (-(-1) - (-1)) \left( \frac{1}{3} (8 - 1) \right) \\
= \frac{14}{3}
\]

\[
M_x = \iint y^2 \, dA \\
= \int_0^\pi \int_0^2 (r \sin \theta)^2 \, r \, d\theta \, dr \\
= \left[ \frac{1}{2} (1 - \cos 2\theta) \right] \left( \frac{1}{4} r^4 \bigg|_{r = 1}^{r = 2} \right) \\
= \frac{15\pi}{8}
\]

\[
\bar{y} = \frac{\frac{15\pi}{8}}{\frac{14}{3}} = \frac{15\pi}{8} \cdot \frac{3}{14} = \frac{45\pi}{112}
\]

Answer:

\( (0, \frac{45\pi}{112}) \)
4. For the curve \( r(t) = \left( \frac{1}{3}t^3 - 7t \right)i + (5t^2)j + \left( \frac{7}{3}t^3 + t \right)k \), find \( a_T \), i.e., the amount of acceleration pointing in the tangential direction of motion. Simplify your answer to a polynomial function of \( t \).

\[
\begin{align*}
\frac{d}{dt}(\|r'(t)\|) &= \frac{d}{dt} \left( \frac{1}{\sqrt{\left( t^2 - 7 \right)^2 + (10t)^2 + (7t^2 + 1)^2}} \right) \\
&= \frac{d}{dt} \left( \frac{1}{5\sqrt{2} (t^2 + 1)} \right) \\
&= 10\sqrt{2}t \\
\end{align*}
\]

METHOD 2

\[
\begin{align*}
r'' &= (2t)i + (10)j + (14t)k \\
r' \cdot r'' &= (t^2 - 7)(2t) + (10t)(10) + (7t^2 + 1)(14t) \\
&= 2t^3 - 14t + 100t + 98t^3 + 14t \\
&= 100t^3 + 100t \\
&= 100t(t^2 + 1) \\
\frac{r' \cdot r''}{\|r'\|^3} &= \frac{100t(t^2 + 1)}{5\sqrt{2}(t^2 + 1)} = \frac{100}{5\sqrt{2}} = \frac{100\sqrt{2}}{10} = 10\sqrt{2}t \\
\end{align*}
\]

Answer: \( 10\sqrt{2}t \)
5. Let $T$ be the solid given by \( \sqrt{x^2 + y^2} \leq z \leq 2 - (x^2 + y^2) \), and let

\[
F = (5x \sin z + \arctan(y))\mathbf{i} + (5y \cos^2 z - \ln(1 + x^2))\mathbf{j} + (7z - e^{x-y})\mathbf{k}.
\]

Use the Divergence Theorem to find \( \iiint_T F \cdot n \, d\sigma \), where $G$ is the surface of $T$ and $n$ is the outward pointing normal vector of the surface.

\[
\begin{align*}
\iiint_T \text{div} \left( \begin{bmatrix}
5x \sin z + \arctan(y) \\
5y \cos^2 z - \ln(1 + x^2) \\
7z - e^{x-y}
\end{bmatrix} \right) \, dV \\
= \iiint_T \left( \frac{5 \sin z}{\partial x} + \frac{5 \cos^2 z}{\partial y} + \frac{7}{\partial z} \right) \, dV \\
= \iiint_T \left( \frac{5 \sin z + 5 \cos^2 z + 7}{5} \right) \, dV \\
= \iiint_T 12 \, dV \\
&= \int_0^{2\pi} \int_0^1 \int_{r=0}^{r=2-r^2} 12r \, dz \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 12r \, dr \int_{z=r}^{z=2-r^2} \, dz \, d\theta \\
&= \int_0^{2\pi} \left[ 6r^2 \right]_0^1 \, d\theta \\
&= 5\theta \bigg|_0^{2\pi} \\
&= 10\pi
\end{align*}
\]

Answer: $10\pi$
6. Find and classify all three critical points of \( h(x, y) = 4x^3 - 6x^2y - 3y^2 + 12y - 7 \).

\[
\begin{align*}
h_x &= 12x^2 - 12xy = 0 \Rightarrow 12x(x-y) = 0 \Rightarrow x = 0 \text{ or } x = y \\
h_y &= -6x^2 - 6y + 12 = 0 \\
&\quad \Rightarrow x = 0 \quad -6y + 12 = 0 \quad \text{so } y = 2 \Rightarrow (0, 2) \\
&\quad \Rightarrow x = y \quad -6x^2 - 6x + 12 = 0 \\
&\quad \quad -6(x^2 + x - 2) = 0 \\
&\quad \quad -6(x+2)(x-1) = 0 \\
&\quad \quad \text{so } x = -2 \text{ or } x = 1 \Rightarrow (-2, 2), (1, 1)
\end{align*}
\]

\[
D = h_{xx} h_{yy} - (h_{xy})^2 = (24x - 12y)(-6) - (-12x)^2
\]

\[
D(0, 2) = 144 \quad \text{and } h_{yy} < 0 \Rightarrow (0, 2) \text{ a local max}
\]

\[
D(-2, -2) = -432 \Rightarrow (-2, -2) \text{ a saddle}
\]

\[
D(1, 1) = -216 \Rightarrow (1, 1) \text{ a saddle}
\]

Answer:

\[
\begin{array}{c|c}
(0, 2) & \text{local max} \\
(-2, -2) & \text{saddle} \\
(1, 1) & \text{saddle}
\end{array}
\]
7. After finishing your final, you head to the theatre to catch the midnight screening of the new Star Wars movie. As you wait to be seated, you decide to pass the time by determining the number of people in line. Your math adrenaline is still pumping from the calculus exam, you decide to do this by using a line integral. In particular, the wall against which people are lined up is given by the curve $C$ with $(x(t), y(t)) = (6t^2, 4\sqrt{3}t^3)$ for $0 \leq t \leq 1$ with distances measured in meters. Also, the number of people in the line at a given $(x, y)$ is $\frac{5}{2}x + \frac{13}{2}$ people per meter. Determine the number of people in line, i.e., find $\int_C \left(\frac{5}{2}x + \frac{13}{2}\right) \, ds$. (Note: Your answer should be a whole number.)

\[ ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \sqrt{(12t)^2 + (12\sqrt{3}t^2)^2} \, dt \]
\[ = \sqrt{144t^2 + 144 \cdot 3t^4} \, dt = \sqrt{144t^2 (1 + 3t^2)} \, dt \]
\[ = 12t \sqrt{1 + 3t^2} \, dt \]

\[ \int_C \left(\frac{5}{2}x + \frac{13}{2}\right) \, ds = \int_0^1 \left(\frac{5}{2}(6t^2) + \frac{13}{2}\right) \cdot 12t \sqrt{1 + 3t^2} \, dt \]
\[ = \int_0^1 \left(30t^2 + 13\right) \sqrt{1 + 3t^2} \, dt \]
\[ = \int_0^1 \left(30t^2 + 13\right) \sqrt{1 + 3t^2} \, du \]
\[ = \int_1^{10u - 10 + 13} u^{\frac{1}{2}} \, du \]
\[ = \left[ \frac{2}{3} u^{\frac{3}{2}} + \frac{3}{2} u^{\frac{3}{2}} \right]_{u=1}^{u=4} \]
\[ = \left(4 \cdot 32 + 2 \cdot 8\right) - \left(4 \cdot 1 + 2 \cdot 1\right) = 128 + 16 - 4 - 2 \]
\[ = 138 \]

Answer: $\boxed{138}$