This test is closed book and closed notes. No sophisticated calculator is allowed for this test. For full credit show all of your work (legibly!). Failure to circle your correct section will result in a 2 point deduction.

1. Let $C(t) = (3\ln(1+(e-1)t^3), 2\cos(\frac{\pi}{2}t))$ for $0 \leq t \leq 1$. Find

$$\int_C \left( \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right).$$

$$M = \frac{\partial f}{\partial x} = 2x + ye^{xy}$$

$$N = \frac{\partial f}{\partial y} = xe^{xy} + c'\left(\frac{\partial f}{\partial y}\right) = 4y + xe^{xy} \Rightarrow c'(y) = 4y$$

$$\Rightarrow c(y) = 2y^2$$

So conservative!!

$$f(x,y) = x^2 + e^{xy} + 2y^2$$

$$C(0) = (3\ln(0), 2\cos(0)) = (0,2) \leftarrow \text{start}$$

$$C(1) = (3\ln(e), 2\cos(\frac{\pi}{2})) = (3,0) \leftarrow \text{end}$$

$$\int_C (2x + ye^{xy}) dx + (4y + xe^{xy}) dy$$

$$= f(3,0) - f(0,2) = (9 + 1 + 0) - (0 + 1 + 0) = 1$$
2. Let $S$ be the set of points satisfying $x^2 + y^2 \leq 1$ and $(x+y)^2 \leq 1$ (see figure). Let $C = \partial S$ be the boundary of the region oriented counter-clockwise. Find

$$\oint_C \frac{(y \cos x - e^{-x} - y) \, dx + (\sin(x) + x + e^y) \, dy}{M \, N}$$

**Green's Theorem**

$$= \iint_S \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA$$

$$= \iint_S \left( \left( \cos(x) + 1 \right) - \left( \cos(x) - 1 \right) \right) \, dA$$

$$= \iint_S 2 \, dA$$

$$= 2 \iint_S \, dA$$

$$= 2 \left( \text{Area of } S \right)$$

$$= 2 \left( \frac{\pi}{2} + 1 \right)$$

$$= \pi + 2$$
3. Let \( \mathcal{J} \) be the solid consisting of points \( x^2 + y^2 + z^2 \leq 4 \) and \( z \geq 0 \). Let \( \partial \mathcal{J} \) denote the bounding surface of \( \mathcal{J} \). Find

\[
\iiint_{\mathcal{J}} \left( \frac{5}{3} x^3 + (y + 1)^2, 5yz^2 + e^y + (x + 1)^2, 5zy^2 - ze^y \right) \cdot \mathbf{n} \, d(SA).
\]

\[
= \iiint_{\mathcal{T}} \left( 5x^2 \right) + (5z^2 + e^y) + (5y^2 e^y) \, dV
\]

\[
= \iiint_{\mathcal{T}} 5(x^2 + y^2 + z^2) \, dV
\]

Region is defined by sphere, so switch to spherical coordinates!

\[
0 \leq \theta \leq 2\pi \\
0 \leq \phi \leq \frac{\pi}{2} \\
0 \leq \rho \leq 2
\]

\[
= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin \phi \, d\phi \int_0^2 5\rho^4 \, d\rho
\]

\[
= \left[ 5\rho^4 \right]_0^2 = 32
\]

\[
= 64\pi
\]
4. Let \( G \) be the surface \((\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\) with \( z \leq 0 \) with normal vectors pointing downward. This is the bottom half of a torus formed by taking a circle of radius 1 and spinning it around the \( z \)-axis with the center distance 2 from the origin in the \( xy \)-plane (see picture; for anyone who likes baking, this surface closely resembles a bundt cake pan). Find

\[
\iint_G (\nabla \times (2xy + 3z + z^2, 2x + x^2 + \cos(z), x + 2xz + e^y)) \cdot n \, d(SA).
\]

\[
\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2xy + 3z + z^2 & 2x + x^2 + \cos z & x + 2xz + e^y
\end{vmatrix}
\]

\[
= (e^y)i + (3z + 2z)j + (2 + 2x)k
\]

\[
= (-\sin z)i - (1 + 2z)j - (2x)k
\]

\[
= \langle e^y + \sin z, 2, 2 \rangle
\]

\[
\iint_H \langle e^y + \sin z, 2, 2 \rangle \cdot \langle 0, 0, -1 \rangle \, d(SA)
\]

\[
= \iint_H (-2) \, d(SA) = (-2)(\text{surface area}) = -2(9\pi - \pi)
\]

\[
= -16\pi
\]
5. Having successfully commandeered the MAV and stripped it of all of its useless equipment (who needs navigational computers or heat shields on rockets anyways), you blast off into orbit as the rescue ship draws near. You are now within sight of the ship, but unfortunately not quite close enough for them to retrieve you. The idea of waving as they pass after working so hard to stay alive is not inviting, so you decide to put a small puncture into your glove to have some of your remaining air escape that will then propel you forward “iron man”-style to bridge the remaining short distance.

Your trajectory can be modeled by the curve in three space \((t \cos(2t), t \sin(2t), \frac{4}{3}t^{3/2})\) for time \(0 \leq t \leq 1\). Given that the amount of oxygen you use at any given point along the curve is \(1 - \frac{9}{16}z^2\) units per distance along the trajectory (i.e., as you start you will use many units but as you get close you will need less), use a line integral to find the number of units of oxygen this short flight uses.

\[
x(t) = t \cos(2t) \\
y(t) = t \sin(2t) \\
z(t) = \frac{4}{3}t^{3/2}
\]

\[
ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \ dt
\]

\[
s = \int_{0}^{1} \sqrt{(\cos(2t) - 2t \sin(2t))^2 + (\sin(2t) + 2t \cos(2t))^2 + (2t^{3/2})^2} \ dt
\]

\[
= \int_{0}^{1} \sqrt{\cos^2(2t) - 4t \cos(2t) \sin(2t) + 4t^2 \sin^2(2t) + \sin^2(2t) + 4t \cos(2t) \sin(2t) + 4t^2 \cos^2(2t) + 4t}
\]

\[
= \int_{0}^{1} \sqrt{1 + 4t + 4t^2} \ dt
\]

\[
= \int_{0}^{1} (1+2t) \ dt
\]

\[
= \left[ t + \frac{t^2}{2} \right]_{0}^{1} = 1 + 1 - \frac{1}{2} - \frac{2}{5}
\]

\[
= \frac{20}{20} + \frac{20}{20} - \frac{5}{20} - \frac{8}{20}
\]

\[
= \frac{27}{20} \text{ units}
\]