Midterm 1 (MATH 265 – Butler)  

This test is closed book and closed notes. No sophisticated calculator is allowed for this test. For full credit show all of your work (legibly!). We note \((t + 1)^3 = t^3 + 3t^2 + 3t + 1\). Each problem is worth 10 points (a total of 50 points). Failure to circle your correct section will result in a 2 point deduction.

1. Rewrite \(\rho = \frac{\cos \phi}{\sin^2 \phi (1 - \sin(2\theta))}\) in Cartesian coordinates and determine when this surface intersects the xy-plane. (It might be helpful to first put this into cylindrical coordinates, and to know \(\sin(2\theta) = 2\sin \theta \cos \theta\).)

\[
\rho \sin^2 \phi \left(1 - \sin(2\theta)\right) = \cos \phi \\
\left(\frac{\rho \sin \phi}{\rho}\right)^2 \left(1 - \sin(2\theta)\right) = \rho \cos \phi \\
\rho \sin^2 \phi \left(1 - 2\sin \theta \cos \theta\right) = \rho \cos \phi \\
Z = r^2 - r^2 \sin(2\theta) = r^2 - 2r^2 \sin \theta \cos \theta \\
= r^2 - 2r \sin \theta \cos \theta \\
\sqrt{x^2 + y^2} \quad \frac{r}{\sqrt{2}} \quad \frac{r \cos \phi}{\sqrt{x^2 + y^2}} \\
= x^2 + y^2 - 2xy \\
= x^2 - 2xy + y^2 \\
= (x - y)^2 \\
\text{intersects when} \quad Z = 0 \quad \text{which only happens when} \quad x - y = 0
\]

\[ Z = (x - y)^2 \]

hits xy-plane when \(x = y\)
2. Let \( \ell_1 \) be the line segment joining \((3, 3, 2)\) with \((-1, 5, 0)\); let \( \ell_2 \) be the line segment joining \((1, 4, 3)\) with \((-3, 2, 1)\); let \( \ell_3 \) be the line segment joining \((4, 7, 6)\) with \((-2, -3, 2)\). Find the unique plane which would cut these three line segments in half (i.e., passes through their midpoints). (This is a special case of what is known as the Ham Sandwich Theorem.)

Plane passes through \((1, 4, 1), (-1, 3, 2)\) \& \((1, 2, 4)\)

Normal vector \( \langle z, 1, -1 \rangle \times \langle z, -1, 2 \rangle = \begin{vmatrix} i & j & k \\ z & 1 & -1 \\ z & -1 & 2 \end{vmatrix} = \langle 2i - 2j - 2k \rangle = \langle 1, -6, -4 \rangle \)

\( x - 6y - 4z = d \)

Plug in \((1, 4, 1)\) \(\rightarrow\)  \(1 - 6 \cdot 4 - 4 \cdot 1 = -27 \Rightarrow x - 6y - 4z = -27\)
3. Find the distance traveled (i.e., length) along the curve \( \frac{x(t)}{a} + \frac{y(t)}{b} + \frac{z(t)}{c} \) between \( t = 0 \) and \( t = 3 \). Simplify your answer as much as possible.

\[
D = \int_{0}^{3} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} \, dt
\]

\[
= \int_{0}^{3} \sqrt{\left( t^{3/2} - t^{-1/2} \right)^2 + (-2t)^2 + (t+1)^2} \, dt
\]

\[
= \int_{0}^{3} \sqrt{t^3 - 2t^2 + t + 4t^2 + t^2 + 2t + 1} \, dt
\]

\[
= \int_{0}^{3} \sqrt{t^3 + 3t^2 + 3t + 1} \, dt
\]

\[
= (t+1)^3 \quad \text{see instructions on page 1}
\]

\[
= \int_{0}^{3} (t+1)^{3/2} \, dt
\]

\[
= \frac{2}{5} (t+1)^{5/2} \bigg|_{t=0}^{t=3}
\]

\[
= \frac{2}{5} \left( 4^{5/2} - \frac{2}{5} \right)
\]

\[
= \frac{2}{5} \cdot 32 - \frac{2}{5} \cdot 1
\]

\[
= \frac{64 - 2}{5} = \frac{62}{5}
\]

\[
= \sqrt{\frac{62}{5}}
\]
4. Find the tangent line to \( r(t) = (e^{2t} - 3t, \sin(2t) - \cos(t), t^2 - 2t + 4) \) at \( t = 0 \). Give your answer in parametric form.

\[
r(t) = (e^{2t} - 3t, \sin(2t) - \cos(t), t^2 - 2t + 4)
\]

\[
r'(t) = (2e^{2t} - 3, 2\cos(2t) + \sin(t), 2t - 2)
\]

\[
r(0) = (1, -1, 4) \quad \text{← position}
\]

\[
r'(0) = (-1, 2, -2) \quad \text{← direction}
\]

\[
\begin{align*}
x &= 1 - t \\
y &= -1 + 2t \\
z &= 4 - 2t
\end{align*}
\]

↑↑ direction
5. (a) For \( t > 0 \), find \( \kappa(t) \) for \( r(t) = \langle \ln t, 2t, t^2 \rangle \).

\[
\begin{align*}
r(t) &= \langle \ln t, 2t, t^2 \rangle \\
r'(t) &= \langle \frac{1}{t}, 2, 2t \rangle \\
r''(t) &= \langle -\frac{1}{t^2}, 0, 2 \rangle
\end{align*}
\]

\[
\begin{align*}
r'(t) \times r''(t) &= \begin{vmatrix} i & j & k \\
\frac{1}{t} & 2 & 2t \\
-\frac{1}{t^2} & 0 & 2 \\
\end{vmatrix} \\
&= 4i - \frac{2}{t}j + 0k \\
&= \langle 4, -\frac{2}{t}, 0 \rangle
\end{align*}
\]

\[
\|r'(t)\| = \sqrt{\left(\frac{1}{t^2}\right)^2 + (2)^2 + (2t)^2} = \sqrt{\frac{1}{t^4} + 4 + 4t^2} = \sqrt{\frac{1}{t^2}(1 + 4t^2 + 4t^4)}
\]

\[
= \sqrt{\frac{1}{t} \left( 1 + 2t^2 \right)^2} = \frac{1}{t} \left( 1 + 2t^2 \right)
\]

\[
\|r'(t) \times r''(t)\| = \sqrt{4^2 + \left( -\frac{2}{t} \right)^2 + \left( \frac{2}{t^2} \right)^2} = \sqrt{16 + \frac{4}{t^2} + \frac{4}{t^4}}
\]

\[
= \sqrt{\frac{4}{t^4} \left( 4t^4 + 4t^2 + 1 \right)} = \frac{2}{t} \left( 1 + 2t^2 \right)
\]

\[
k(t) = \frac{\frac{2}{t^2} \left( 1 + 2t^2 \right)}{\left( \frac{1}{t} \left( 1 + 2t^2 \right) \right)^3} = \frac{\frac{2}{t^2} \left( 1 + 2t^2 \right)}{\frac{1}{t^3} \left( 1 + 2t^2 \right)^3} = \frac{2t}{\left( 1 + 2t^2 \right)^2}
\]

(b) Determine the \( t \) which maximizes \( \kappa(t) \) found in part (a).
(It suffices to find \( t \), you do not need to prove why it is maximal.)

Maximized when \( k'(t) = 0 \)

\[
k'(t) = \frac{(1+2t^2)^2 \cdot 2t - 2t \cdot 2(1+2t^2) \cdot 4t}{(1+2t^2)^2} = 0
\]

\[
= 2 \cdot (1+2t^2) \left[ (1+2t^2) - 8t^2 \right] = 0
\]

\[
= 2 \cdot (1+2t^2) \left[ 1 - 6t^2 \right] = 0
\]

\[
\Rightarrow t^2 = \frac{1}{6} \quad \text{or} \quad t = \frac{1}{\sqrt{6}}
\]
This test is closed book and closed notes. No sophisticated calculator is allowed for this test. For full credit show all of your work (legibly!). The volume of a cone is $\frac{1}{3}(\text{base})(\text{height})$. Each problem is worth 10 points (a total of 50 points). Failure to circle your correct section will result in a 2 point deduction.

1. The point $(2, 1, 3)$ is the midpoint of the points $P$ and $Q$. If the point $P$ lies on the line $\langle x, y, z \rangle = \langle 5, 2, 4 \rangle + t\langle 2, -3, 1 \rangle$, then $Q$ lies on some other line. Determine this other line and give it in parametric form.

$$Q = (x, y, z)$$

$$\text{midpoint} = (2, 1, 3) = \left( \frac{x + (5+2t)}{2}, \frac{y + (2-3t)}{2}, \frac{z + (4+t)}{2} \right)$$

$$(5+2t, z-3t, 4+t)$$

\[
\begin{align*}
    x + (5+2t) &= 4 \\
    y + (2-3t) &= 2 \\
    z + (4+t) &= 0
\end{align*}
\]

\[
\begin{align*}
    x &= -1 - 2t \\
    y &= 3t \\
    z &= z - t
\end{align*}
\]
2. Sketch the region and find the volume of all points satisfying $z - 1 \leq r \leq \frac{1}{3}z + 1$. (Hint: rewrite inequalities in terms of $z$, what shape(s) bound this region?)

$\frac{2}{3}z + 1 \leq r$ or $z \leq r + 1 \leftarrow$ cone shifted up by 1

$\frac{1}{3}z + 3 \geq r - 1$ or $z \geq 3r - 3 \leftarrow$ cone stretched and shifted down by 3

Region is between these two cones

$r + 1 = 3r - 3$ or $2r = 4$

Intersection if

$or$ $r = 2$

Corresponds to $z = 3$

Volume of cone $= \frac{1}{3} \cdot \text{(base)} \cdot \text{(height)}$

Volume = $\frac{1}{3} \cdot \pi \cdot z \cdot 1 - \frac{1}{3} \cdot \pi \cdot z \cdot 1$

= $\frac{1}{3} \cdot \pi \cdot z^2 \cdot 6 - \frac{1}{3} \cdot \pi \cdot z^2 \cdot 2$

= $\frac{24}{3} \pi - \frac{8}{3} \pi$

= $\frac{16}{3} \pi$
3. Find the distance traveled along the curve \((\frac{1}{3} t^5 - \frac{1}{2} t^4 + 1, \frac{\sqrt{3}}{4} t^4 + 3, \frac{4}{9} t^{9/2} + 7)\) between \(t = 0\) and \(t = 4\). (You do not have to simplify after evaluating the integral; but it does simplify.)

\[
D = \int_0^4 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt
\]

\[
= \int_0^4 \sqrt{(t^4 - 2t^3)^2 + (\frac{\sqrt{3}}{4} t^4)^2 + (\frac{4}{9} t^{9/2})^2} \, dt
\]

\[
= \int_0^4 \sqrt{t^8 - 4t^7 + 4t^6 + 5t^6 + 4t^4} \, dt
\]

\[
= \int_0^4 \sqrt{t^8 + 9t^6} \, dt
\]

\[
= \int_0^4 t^6 (t^2 + 9) \, dt
\]

\[
= \int_0^4 t^6 \sqrt{u} \, du
\]

\[
= \int_0^{t^2 + 9} \frac{1}{2} u^{1/2} \, du
\]

\[
= \frac{1}{2} \int_0^{t^2 + 9} \frac{1}{2} u^{1/2} \, du
\]

\[
= \left[ \frac{1}{5} u^{5/2} - 3u^{3/2} \right]_{u=25}^{u=9}
\]

\[
= \left( \frac{1}{5} 25^{5/2} - 3 \cdot 25^{3/2} \right) - \left( \frac{1}{5} \cdot 9^{5/2} - 3 \cdot 9^{3/2} \right)
\]

\[
= \left( \frac{1}{5} \cdot 25^{5/2} - 3 \cdot 25^{3/2} \right) - \left( \frac{1}{5} \cdot 9^{5/2} - 3 \cdot 3^{3} \right)
\]

\[
= \left( \frac{1}{5} \cdot 5^5 - 3 \cdot 5^3 \right) - \left( \frac{243}{5} - 81 \right)
\]

\[
= 331 \cdot \frac{243}{5} = \frac{1655}{5} - \frac{243}{5} = \frac{1412}{5}
\]

\[\text{OK to stop here}\]
4. Find the osculating plane to \( r(t) = \langle e^{2t} - 3t, \sin(2t) - \cos(t), t^2 - 2t + 4 \rangle \) at \( t = 0 \). (Recall the osculating plane contains the point, the direction of motion, and the direction of acceleration.)

So we can use \( r'(t) \times r''(t) \)

\[
\begin{align*}
r(t) &= \langle e^{2t} - 3t, \sin(2t) - \cos(t), t^2 - 2t + 4 \rangle \\
r'(t) &= \langle 2e^{2t} - 3, 2\cos(2t) + \sin(t), 2t - 2 \rangle \\
r''(t) &= \langle 4e^{2t}, -4\sin(4t) + \cos(t), 2 \rangle \\
r(0) &= \langle 1, -1, 4 \rangle \quad \text{point on plane} \\
r'(0) &= \langle -1, 2, -2 \rangle \\
r''(0) &= \langle 4, 1, 2 \rangle
\end{align*}
\]

\[
r'(0) \times r''(0) = \begin{vmatrix} i & j & k \\ -1 & 2 & -2 \\ 4 & 1 & 2 \end{vmatrix} = 4i - 8j - k \\
= 6i - 6j - 9k \\
= \langle 6, -6, -9 \rangle
\]

For convenience, we can scale by \( \frac{1}{3} \)

\[
2x - 2y - 3z = -8
\]

Plug in point \((1, -1, 4)\)

\[
2x - 2y - 3z = -8
\]
5. (a) For \( t > 0 \), find \( a_T \) (i.e., the amount of acceleration pointing in the direction of motion) for \( r(t) = \langle \ln t, 2t, t^2 \rangle \).

\[
\frac{r'(t) \cdot r''(t)}{\|r'(t)\|}
\]

\[
r(t) = \langle \ln t, 2t, t^2 \rangle
\]
\[
r'(t) = \langle \frac{1}{t}, 2, 2t \rangle
\]
\[
r''(t) = \langle -\frac{1}{t^2}, 0, 2 \rangle
\]

\[
\|r'(t)\| = \sqrt{\left(\frac{1}{t}\right)^2 + (2)^2 + (2t)^2} = \sqrt{\frac{1}{t^2} + 4 + 4t^2} = \sqrt{\frac{1}{t^2}(1 + 4t^2 + 4t^4)}
\]

\[
= \sqrt{\frac{1}{t^2}(1 + 2t^2)^2} = \frac{1}{t}(1 + 2t^2)
\]

\[
r'(t) \cdot r''(t) = \langle \frac{1}{t}, 2, 2t \rangle \cdot \langle -\frac{1}{t^2}, 0, 2 \rangle = -\frac{1}{t^3} + 4t = \frac{1}{t^3}(4t^4 - 1)
\]

\[
a_T = \frac{\frac{1}{t^3}(4t^4 - 1)}{\frac{1}{t}(1 + 2t^2)} = \frac{4t^4 - 1}{t^2(1 + 2t^2)}
\]

(b) Use part (a) to determine the \( t \) when acceleration is perpendicular to motion.

\( a_T = 0 \) means

\[
\frac{4t^4 - 1}{t^2(1 + 2t^2)} = 0 \implies 4t^4 - 1 = 0
\]

\[
\implies t^4 = \frac{1}{4}
\]

\[
\implies t = \sqrt[4]{\frac{1}{4}} = \frac{1}{\sqrt[4]{4}} = \frac{1}{\sqrt{2}}
\]