1. (a) Show that the number of labelled trees on \( n \) vertices where vertex \( i \) has degree \( d_i \) is
\[
\frac{(n - 2)!}{(d_1 - 1)! (d_2 - 1)! \cdots (d_n - 1)!}.
\]
(b) Using the multinomial theorem show that part (a) implies Cayley’s Theorem that the number of labelled trees is \( n^{n-2} \).

(Note: part (a) can be proved easily using Prüfer codes or directly, the latter case along with (b) gives another proof of Cayley’s Theorem.)

2. A tournament on \( n \) vertices is a directed graph which is formed by taking the edges of \( K_n \) and assigning each edge between a pair of vertices, \( u \) and \( v \), one of the two possible orientations, i.e., \( u \leftarrow v \) or \( u \rightarrow v \). In particular there are \( 2^{\binom{n}{2}} \) such graphs on \( n \) vertices.

Show that in each tournament there is a directed path that visits each vertex exactly once.

3. Let \( G \) be a multi-graph, i.e., a graph which can have multiple edges between pairs of vertices. Given an edge \( e = \{u, v\} \), let \( G - e \) be the graph resulting from the deletion of the edge \( e \) from \( G \) and let \( G \setminus e \) be the graph resulting from removing \( e \) and then identifying (i.e., merging) vertices \( u \) and \( v \) together into a new vertex \( w \) and then adding an edge \( x \sim w \) for each edge \( x \sim u \) or \( x \sim v \).

If \( t(G) \) is the number of spanning trees of \( G \), show that
\[
t(G) = t(G - e) + t(G \setminus e).
\]

4. Recall that given two graphs \( G = (V(G), E(G)) \) and \( H = (V(H), E(H)) \) the Cartesian product is \( G \Box H \) which has vertex set \( V(G) \times V(H) = \{(u, v) : u \in V(G), v \in V(H)\} \) and an edge \( (u, v) \sim (x, y) \) if and only if either \( (u = x \text{ and } v = y) \) or \( (u \sim x \text{ and } v = y) \). Given that \( G \) and \( H \) are non-empty graphs show the following:
- \( G \Box H \) is isomorphic to \( H \Box G \).
- \( G \Box H \) is bipartite if and only if both \( G \) and \( H \) are bipartite.
- \( G \Box H \) is connected if and only if both \( G \) and \( H \) are connected.

5. Show that every graph \( G \) on \( n \) vertices has a bipartite subgraph \( H \) where
\[
|E(H)| \geq \frac{|E(G)|}{2}.
\]

(Hint: this can be done using induction on \( n \).)