This test is closed book and closed notes. No sophisticated calculator is allowed for this test. For full credit show all of your work (legibly!). Each problem is worth 10 points (a total of 50 points). Failure to write your correct section will result in a 2 point deduction.

1. Rewrite the following as a single integral by changing the order of integration.

\[ \int_{-2}^{0} \int_{\frac{1}{2} y-1}^{\frac{1}{2} \sqrt{y+1}} f(x, y) \, dx \, dy + \int_{0}^{6} \int_{\frac{1}{2} y-1}^{\frac{1}{2} \sqrt{y+1}} f(x, y) \, dx \, dy. \]

\[ x = \pm \sqrt{\frac{1}{2} y+1} \]
\[ x^2 = \frac{1}{2} y + 1 \]
\[ x^2 - 1 = \frac{1}{2} y \]
\[ y = 2x^2 - 2 \]

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\[ \int_{-1}^{2} \int_{2x^2 - 2}^{2x+2} f(x, y) \, dy \, dx \]
2. Find the center of mass for the lamina which lies between the circle of radius 1 centered at (1, 0) and the circle of radius 2 centered at (2, 0) given that \(\delta(x, y) = x\). (See image to the right.)

The following might be helpful: 
\[
\begin{align*}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t \, dt &= \frac{1}{2} \pi, \\
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt &= \frac{1}{2} \pi, \\
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 t \, dt &= \frac{3}{8} \pi, \\
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 t \, dt &= \frac{3}{8} \pi, \\
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^6 t \, dt &= \frac{5}{16} \pi, \\
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6 t \, dt &= \frac{5}{16} \pi
\end{align*}
\]

\[y = 0 \text{ by symmetry}\]

\[
\bar{x} = \frac{M_y}{M} = \frac{\iint_R x \delta(x, y) \, dA}{\iint_R \delta(x, y) \, dA}
\]

\[
M = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r \cos \theta \cdot r \, dr \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ \frac{1}{3} r^3 \cos \theta \right]_{r=1}^{r=2} \, d\theta
\]

\[
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{1}{3} (64 \cos \theta - \frac{8}{3} \cos^3 \theta) \right) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{56}{3} \cos \theta \, d\theta
\]

\[
= \frac{56}{3} \cdot \frac{8}{6} = \frac{7}{3} \pi
\]

\[
M_y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \cos^2 \theta \cdot r \, dr \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ \frac{1}{4} r^4 \cos^2 \theta \right]_{r=1}^{r=2} \, d\theta
\]

\[
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (64 \cos^6 \theta - 64 \cos^6 \theta) \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 60 \cos^6 \theta \, d\theta
\]

\[
= \frac{15}{16} \cdot \frac{56}{3} \pi = \frac{75}{4} \pi
\]

\[
\bar{x} = \frac{75 \pi}{\frac{75}{28}} = \frac{75 \pi \cdot \frac{7}{75}}{7 \pi} = \frac{75}{28}
\]
3. Rewrite the following integral in terms of cylindrical coordinates and then evaluate:

\[ \int_0^2 \int_0^{4-y^2} \int_0^{\sqrt{4-y^2-z}} \frac{1}{4-x^2-y^2} \, dx \, dz \, dy. \]

\[ x = \sqrt{4-y^2-z} \]
\[ x^2 = 4-y^2-z \]
\[ z = 4 - x^2 - y^2 \]
\[ \text{upside down bowl} \]

Region is one-fourth of an upside down bowl:

\[ 0 \leq z \leq 4-x^2-y^2 \]
\[ 0 \leq r \leq 2 \]
\[ 0 \leq \theta \leq \frac{\pi}{2} \]

From above (i.e. z-axis):

\[
\int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{4-r^2} \frac{1}{4-r^2} \, r \, dz \, dr \, d\theta
\]

\[ = \int_0^{\frac{\pi}{2}} \int_0^2 \left. \frac{r \, z}{4-r^2} \right|_{z=0}^{z=4-r^2} \, dr \, d\theta \]

\[ = \int_0^{\frac{\pi}{2}} \int_0^2 r \, dr \, d\theta \]

= Area of quarter circle of radius 2

\[ = \pi \]
4. Use the substitutions \( u = x \) and \( v = y + x^3 \) to rewrite the following integral in terms of \( u \) and \( v \) and use this to calculate the integral.

\[
\int_0^1 \int_{-x^3}^{x-x^3} 20(y+x^3)^3 e^{x^3} \, dy \, dx
\]

\[
J(u, v) = \begin{vmatrix} 1 & 0 \\ -3u & 1 \end{vmatrix} = 1
\]

\[
\int_0^1 \int_0^u 20v \, dv \, du
\]

\[
\int_0^1 5u \, du = \frac{5}{2} \left. \left( u^2 \right) \right|_0^1 = \frac{5}{2}
\]

\[
\int_0^1 e^u \, du = \left. e^u \right|_0^1 = e - 1
\]
5. Find the following:

\[ \int_{-1}^{1} \int_{0}^{2} \left( e^{y^4 + y^2} \sin(x^{137}) + (y - 1)^9 \arctan(x^4 + 16 \cos x) + 2 \right) \, dy \, dx = 0 + 0 + 8 = 8 \]

(Hint: break up into three integrals and work on each part separately.)

\[ \int_{-1}^{1} \int_{0}^{2} e^{y^4 + y^2} \sin(x^{137}) \, dy \, dx = \int_{0}^{2} \int_{-1}^{1} e^{y^4 + y^2} \sin(x^{137}) \, dx \, dy = 0 \]

\[ \text{Odd function w.r.t. } x \text{ so integral is 0} \]

\[ \int_{-1}^{1} \int_{0}^{2} (y - 1)^9 \arctan(x^4 + 16 \cos x) \, dy \, dx \]

\[ = \int_{-1}^{1} \arctan(x^4 + 16 \cos x) \left. \frac{1}{10} \left( y - 1 \right)^{10} \right|_{y=0}^{y=2} \, dx \]

\[ = \int_{-1}^{1} \arctan(x^4 + 16 \cos x) \left. \left( \frac{1}{10} \left( 0 - (-1)^{10} \right) \right) \right|_{x=0}^{x=1} \, dx = 0 \]

\[ \int_{-1}^{1} \int_{0}^{2} 2 \, dy \, dx = \text{twice this area} = 8 \]