Lines

To find a line we will need two items: a point on the line, \((x_0, y_0, z_0)\); and a direction vector for the line, \((a, b, c)\). Once we have these we can write an equation for the line. There are three different ways we have discussed to write such an equation.

The first is vector format where we find all the points (in vector form) which can be done by starting at the given point and adding some multiple of the direction vector, i.e.,

\[
\begin{align*}
\langle x, y, z \rangle &= \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle.
\end{align*}
\]

Solving for each component we get parametric form, i.e., find a parametric equation for the line.

\[
\begin{align*}
x &= x_0 + at \\
y &= y_0 + bt \\
z &= z_0 + ct
\end{align*}
\]

If we solve for \(t\) in parametric form we can get an expression for the line only in terms of \(x\), \(y\) and \(z\) known as symmetric form, i.e.,

\[
\begin{align*}
\frac{x - x_0}{a} &= \frac{y - y_0}{b} = \frac{z - z_0}{c}.
\end{align*}
\]

This assumes that \(a, b, c \neq 0\), if one of them is 0 then we simply write this as a combination of equations. For example if \(a = 0\) then the line would be

\[
\begin{align*}
x &= x_0 \\
y &= y_0 + bt \\
z &= z_0 + ct
\end{align*}
\]

In each of the forms given above it is easy to find a point on the line and the direction vector of the line. We note that these equations are not unique for lines, i.e., we can choose a different point or choose a parallel vector and still have the same line.

Given a parametric curve \((x(t), y(t), z(t))\) we can transform it to a vector valued function by drawing a vector from the origin to the current location on the parametric curve, i.e., \(r(t) = \langle x(t), y(t), z(t) \rangle\). Recall that velocity is given by \(v(t) = r'(t)\). Velocity encodes both the speed of the particle and the current direction in which the particle is moving.

If we imagine the particle as traveling along a track and at some time \(t_0\) we derail from the track, then how does the particle move? Well it starts at the point it derailed at, \(r(t_0)\), it will move in the direction given by velocity, \(r'(t_0)\). This movement will thus lie on the line containing the point \(r(t_0)\) and the direction vector \(r'(t_0)\). This is known as the tangent line.

Decomposing acceleration

We can continue to refine our conversation about motion. If we only care about the direction that the particle is traveling in (and not the speed), then we would naturally want to consider a unit vector in the direction of movement. We have the following.

\[
\begin{align*}
r(t) &= \text{position} \\
v(t) &= r'(t) = \text{velocity} \\
a(t) &= r''(t) = \text{acceleration} \\
T(t) &= \frac{r'(t)}{\|r'(t)\|} = \text{unit tangent vector} \\
N(t) &= \frac{T'(t)}{\|T'(t)\|} = \text{unit normal vector}
\end{align*}
\]

The unit tangent vector is pointing in the direction of motion. Since \(\|T\| = 1\) (i.e., it is unit) we have using a result from last week that \(T\) is perpendicular to \(T'\), and hence \(T\) is perpendicular to \(N\). This indicates that \(N\) is perpendicular to the direction vector \(T\), that is to say perpendicular to the motion of the particle. On the other hand since \(T\) is related to velocity we would expect that \(T'\) should be related to acceleration, i.e., \(N\) should have something to do with acceleration. This turns out to be the case. In particular we have the following:

\[
\begin{align*}
a &= a_T T + a_N N \\
a_T &= \frac{r' \cdot r''}{\|r'\|} \quad \text{and} \quad a_N = \frac{\|r' \times r''\|}{\|r'\|}
\end{align*}
\]

Which is to say that we can split acceleration into two parts; one part in the direction the particle is currently moving, and another in a direction orthogonal to how the particle is currently moving. The first term is essentially the projection of \(r''\) onto \(r'\), i.e., we have

\[
\text{proj}_{r'}(r'') = \frac{r' \cdot r''}{\|r'\|^2} r' = \frac{r' \cdot r''}{\|r'\|} \|r'\| = \frac{r' \cdot r''}{\|r'\|} T = a_T T.
\]

The other term essentially is a projection onto the orthogonal direction which can be done by using cross products. Alternatively, we can make the observation that \(a_N N = a - a_T T\).

From the perspective of the particle then all of the interesting information about the motion (position, velocity and acceleration) are all contained in the single plane containing the point and the vectors \(T\) and \(N\). The normal vector to this plane is known as the binormal vector and is denoted \(B = T \times N\). This plane is known as the osculating plane (or kissing plane, i.e., it gently kisses the curve).

Curvature

Given a curve we can ask how “bendy” the curve is. Which is to say we want to measure how fast the curve is turning. We are careful here to say that we are not interested in our speed of rotation in regards
to how we travel along the curve, i.e., this is not dependent on the parameterization.  

A first approach is to look at how quickly our direction, $T$, is changing as we change our position along the curve, $s$. So curvature (which is denoted with the Greek letter $\kappa$) is given by

$$\kappa = \frac{\left\| \frac{dT}{ds} \right\|}{\left\| r' \right\|}.$$

This is great, but is hard to compute for most curves. So using the chain rule we can rewrite this as

$$\kappa = \frac{\left\| T' \right\|}{\left\| r' \right\|^3}.$$

This is better but we can even make it more straightforward (i.e., $T$ tends to have square roots which are best to avoid when taking derivatives if we can). This is done by recalling that $\left\| T' \right\|$ shows up as part of $a_N$. Using this we derive

$$\kappa = \frac{|x'y'' - x''y'|}{\left( (x')^2 + (y')^2 \right)^{3/2}}.$$

**Quadric surfaces**

To understand surfaces we will often look at cross sections. These are found by looking at where a plane intersects the surface and examining the resulting curve(s). We will be particularly interested in *traces* which correspond to planes of the form $x = c$, $y = c$ or $z = c$. Note that in such planes one of the variables is fixed so any equation describing a surface reduces the number of equations involved.

A special type of surface is a *cylinder*. These are surfaces which have identical traces in one of the variables, i.e., $x^2 + y^2 = 1$ in three dimensions forms what we normally call a cylinder because for each slice of the form $z = c$ we get a unit circle. These are easy to identify when written out as equations because they are missing a variable.

With our background in understanding conic sections (i.e., ellipses, parabolas, hyperbolas and so forth) we are ready to understand what is going on for cross sections of quadric surfaces. These are surfaces which can be written in the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Flyz + Gx + Hy + Iz + J = 0,$$

where $A, B, \ldots, J$ are constants. But by translation and rotation we only need to consider such surfaces of the form

$$Ax^2 + By^2 + Cz^2 + J = 0 \text{ or } Ax^2 + By^2 + Iz = 0.$$

These give the following surfaces:

- **Ellipsoids**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
  Cross sections: ellipses, empty
- **Hyperboloid of one sheet**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
  Cross sections: ellipses, hyperbolas
- **Hyperboloid of two sheets**: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
  Cross sections: ellipses, hyperbolas
- **Elliptic paraboloid**: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
  Cross sections: ellipses, parabolas, empty
- **Hyperbolic paraboloid**: $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$
  Cross sections: parabolas, hyperbolas
- **Elliptic cone**: $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
  Cross sections: ellipses, hyperbolas

**Review problems (one to appear on Quiz 4)**

1. Find the point on the plane $x + 2y = 5 + 3z$ closest to the point $(4, 4, -7)$. (Hint: the line between these two points is perpendicular to the plane.)

2. While holding an egg you travel along

$$r(t) = (t^3 + t)i + (6t^2 - 7)j + (7 - t^3)k.$$

At time $t = 1$ you let go of the egg. Determine where the egg will hit the xy-plane.

3. Find the line in parametric form containing the point $(0, 1, 2)$ and which perpendicularly intersects the line $x = 1 + t$, $y = 1 - t$, $z = 2t$.

4. Let $r(t) = (2\cos(\pi t), \sin(\pi t), t^3)$. Find the plane perpendicular to this curve at time $t = 2$.

5. Find $a_T$ and $a_N$ for $r(t) = ti + t^2j + \frac{t}{2}k$. Simplify your answers.

6. Find curvature $k(t)$ for $r(t) = (\sin t, \cos t, \frac{1}{2}t^2)$.

7. Find the curvature at time $t = 0$ for the curve

$$r(t) = \left(71 + \ln(t + 1), \tan t + 2t^2 + \sqrt{\pi}, e^t \cos t - \frac{e^t}{6}\right).$$

8. Give a formula for, and classify, the quadric surface containing the parametric curve

$$(e^t \sin t + \cos t), e^t, e^{2t} \sin(2t)).$$