1. Find the directional derivative of \( f(x, y, z) = xz^2 - 3xy + 2xyz - 3x + 5y - 17 \) from the point \((2, -6, 3)\) in the direction of the origin.

To find a directional derivative we note that \( D_u f = \nabla f \cdot u \). So we need to find \( u \) and \( \nabla f \). Since we are going from the point \((2, -6, 3)\) towards \((0, 0, 0)\) then a vector pointing in the appropriate direction is \((-2, 6, -3)\). This is not a unit vector since \( \|(-2, 6, -3)\| = \sqrt{4 + 36 + 9} = \sqrt{49} = 7 \), but by scaling we can make it a unit vector so that \( u = \frac{1}{7}(-2, 6, -3) \). The gradient is

\[
\nabla f(x, y, z) = \langle z^2 - 3y + 2yz - 3, -3x + 2xz + 5, 2xz + 2xy \rangle.
\]

Evaluating at the point \((2, -6, 3)\) we have

\[
\nabla f(2, -6, 3) = \langle (2^2) - 3(-6) + 2(-6)3 - 3, -3\cdot2 + 2\cdot2\cdot3 + 5, 2\cdot2\cdot3 + 2\cdot2(-6) \rangle = \langle -12, 11, -12 \rangle.
\]

Therefore the desired directional derivative will be given by

\[
\nabla f \cdot u = \langle -12, 11, -12 \rangle \cdot \frac{1}{7}(-2, 6, -3) = \frac{24 + 66 + 36}{7} = \frac{126}{7} = 18.
\]
2. Find $\frac{\partial f}{\partial x}(0, 1)$ and $\frac{\partial f}{\partial y}(0, 1)$ for

$$f(x, y) = \sin x + y^2 \cos x + y^4 \arctan \left(x(y^2 - 1)\right) + \ln(2e^{\sin x} - 1) \sec(xy) \tan(y - 1).$$

(Hint: $(\partial f/\partial x)(a, b) = g'(a)$ where $g(x) = f(x, b)$, similarly for $(\partial f/\partial y)(a, b).$)

Interpreting the hint it says that whatever variable we are not differentiating with respect to we can first plug in the value, simplify and then differentiate. We have,

$$g(x) = f(x, 1) = \sin x + \cos x$$

(the rest of the terms are 0). Now differentiating with respect to $x$ we have

$$g'(x) = \cos x - \sin x$$

and so

$$\frac{\partial f}{\partial x}(0, 1) = g'(0) = 1.$$

Similarly,

$$h(y) = f(0, y) = y^2$$

(the rest of the terms are 0). Now differentiating with respect to $y$ we have

$$h'(y) = 2y$$

and so

$$\frac{\partial f}{\partial y}(0, 1) = h'(1) = 2.$$
3. Evaluate the following limits or show that they do not exist. Briefly(!) justify your answer.

(a) \[ \lim_{(x,y) \to (0,0)} \frac{x \sin x}{x^2 + y^4} = \]

Initially when we plug in (0, 0) we get 0/0 so this means that we have work to do! Let us see what happens as we approach along different axis. Along the x-axis (i.e., \( y = 0 \)) we have

\[ \lim_{(x,0) \to (0,0)} \frac{x \sin x}{x^2 + 0^4} = \lim_{x \to 0} \frac{\sin x}{x} = 1. \]

(One of our favorite limits of all time! If we didn’t remember it we could also have used L’Hospital’s rule.)

Along the y-axis (i.e., \( x = 0 \)) we have

\[ \lim_{(0,y) \to (0,0)} \frac{0 \sin 0}{0^2 + y^4} = \lim_{y \to 0} 0 = 0. \]

These don’t match! Therefore the limit does not exist.

(b) \[ \lim_{(x,y) \to (1,1)} \frac{x \sqrt{|y|} \sin x}{x^2 + y^4} = \]

This looks similar to the first limit. But there is an important difference, and it is not the \( \sqrt{|y|} \), it is that we are going to (1, 1). In particular, this function is made up of continuous parts and the denominator is \( \text{not zero at} \) (1, 1) and so to evaluate the limit all we need to do is plug in (1, 1), so that

\[ \lim_{(x,y) \to (1,1)} \frac{x \sqrt{|y|} \sin x}{x^2 + y^4} = \frac{\sin 1}{2}. \]

(On a side note the limit of this function at the origin \textit{does} exist since we can bound this above and below by \( \pm \sqrt{|y|} \). But this has nothing to do with the problem.)
4. Let \( f(x, y) = x^3 - 8xy + 2y^2 - 3x + 4y - 23 \).

(a) Find the two critical points for \( f(x, y) \).

To find the critical points we solve

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 3x^2 - 8y - 3 = 0 \\
\frac{\partial f}{\partial y} &= -8x + 4y + 4 = 0
\end{align*}
\]

Taking the second equation and solving for \( y \) we have \( y = 2x - 1 \), which if we substitute into the first equation we have

\[
3x^2 - 8(2x - 1) - 3 = 3x^2 - 16x + 5 = (3x - 1)(x - 5) = 0.
\]

So that \( x = 1/3 \) or \( x = 5 \). Now using \( y = 2x - 1 \) we get that our two critical points are at \( \left( \frac{1}{3}, -\frac{1}{3} \right) \) and \( (5, 9) \).

(b) Determine if these points are maximums, minimums or neither.

Calculating the second derivatives we have \( f_{xx} = 6x \), \( f_{yy} = 4 \) and \( f_{xy} = -8 \), so that

\[
D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 24x - 64.
\]

Plugging in we have

\[
D\left( \frac{1}{3}, -\frac{1}{3} \right) = -56 < 0 \quad \text{so} \quad \left( \frac{1}{3}, -\frac{1}{3} \right) \quad \text{is a saddle, and}
\]

\[
D(5, 9) = 56 > 0 \quad \text{and} \quad f_{xx}(5, 9) = 30 > 0 \quad \text{so} \quad (5, 9) \quad \text{is a minimum.}
\]
5. Find the tangent plane to \( f(x, y) = x^3 y - 3xy^2 \) at \((2, 1)\).

The equation for the tangent plane is given by

\[
z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).
\]

We already know that \((a, b) = (2, 1)\) and that \(f(2, 1) = 2\) so we need the partial derivatives. We have

\[
\frac{\partial f}{\partial x}(x, y) = 3x^2 y - 3y^2, \text{ and } \\
\frac{\partial f}{\partial y}(x, y) = x^3 - 6xy,
\]

so that

\[
\frac{\partial f}{\partial x}(2, 1) = 9, \text{ and } \\
\frac{\partial f}{\partial y}(2, 1) = -4.
\]

Substituting all of the values in we get our tangent plane

\[
z = 2 + 9(x - 2) - 4(y - 1) \quad \text{or} \quad z = 9x - 4y - 12
\]