THE MEAN VALUE THEOREM

This note presents a theorem of fundamental importance sometimes known as the Mean Value Inequality.

**Theorem 1** (Mean Value Theorem). Let \( U \subset \mathbb{R}^n \) be open, and let \( f : U \to \mathbb{R}^m \) be a \( C^1 \) function. Let \( a, b \in U \) and assume further that the segment \( \{a + t(b - a) : 0 \leq t \leq 1\} \subset U \). Let
\[
M = \max_{0 \leq t \leq 1} \|Df(a + t(b - a))\|.
\]
Then
\[
\|f(b) - f(a)\| \leq M \|b - a\|.
\]

**Proof.** It suffices to prove that for every \( \varepsilon > 0 \) the inequality (1) is true when \( M \) is replaced by \( M + \varepsilon \).

Let \( \varepsilon > 0 \) be given.

Call \( t \in [0, 1] \) an \( S \)-point if for every \( s \) with \( 0 \leq s < t \),
\[
\|f(a + s(b - a)) - f(a)\| \leq (M + \varepsilon) s \|b - a\|
\]
Note that if \( t \) is an \( S \)-point, then
- (a) every \( t' \) with \( 0 \leq t' < t \) is also an \( S \)-point;
- (b) inequality (2) holds for \( s = t \), by the continuity of \( f \).

Let \( S \) be the set of \( S \)-points. We will prove that \( S = [0, 1] \). This is obviously a stronger statement than the Theorem.

We begin by proving that \( S \) is not empty.

Since \( f \) is differentiable at \( a \), there exists \( \delta > 0 \) such that whenever \( \|h\| < \delta \),
\[
\|f(a + h) - f(a)\| \leq \|Df(a) \cdot h\| + \varepsilon \|h\|
\leq \|Df(a)\| \cdot \|h\| + \varepsilon \|h\|
\leq (M + \varepsilon) \|h\| \quad (\text{Since } \|Df(a)\| \leq M.)
\]
This proves that \( \delta_0 = \delta/\|b - a\| \) is an \( S \)-point: for any \( s \) with \( 0 \leq s < \delta_0 \) we take \( h = s(b - a) \) in the inequality just proved, and derive (2).

The proof is now complete: by the remarks (a) and (b) above, \( S \) is a closed subinterval of \([0, 1]\) containing 0. No \( t < 1 \) is an upper bound for \( S \) because we can apply the argument just given to show that whenever \( t < 1 \) and \( t \in S \) there is some \( \delta_0 > 0 \) so that \( t + \delta_0 \in S \). \( \square \)