Submultiplicativity of the numerical radius of commuting matrices of order two

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Dedicated to Professor Pei Yuan Wu.

Abstract
Denote by $w(T)$ the numerical radius of a matrix $T$. An elementary proof is given to the fact that $w(AB) \leq w(A)w(B)$ for a pair of commuting matrices of order two, and characterization is given for the matrix pairs that attain the quality.

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1 Introduction
Let $M_n$ be the set of $n \times n$ matrices. The numerical range and numerical radius of $A \in M_n$ are defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\} \quad \text{and} \quad w(A) = \max\{||\mu|| : \mu \in W(A)\},$$

respectively. The numerical range and numerical radius are useful tools in studying matrices and operators; for example, see [3, 4, 7]. It is known that $w(A)$ is a norm on $M_n$ with many interesting properties; for example, see [3, 4, 5, 6, 7]. For example,

$$w(A) \leq \|A\| \leq 2w(A),$$

where $\|A\| = \max\{(x^*A^*Ax)^{1/2} : x \in \mathbb{C}^n, x^*x = 1\}$ is the operator norm of $A$. While the spectral norm is submultiplicative, i.e., $\|AB\| \leq \|A\|\|B\|$ for all $A, B \in M_n$, the numerical radius is not. In general,

$$w(AB) \leq \xi w(A)w(B) \quad \text{for all } A, B \in M_n$$

if and only if $\xi \geq 4$; e.g., see [2]. Despite the fact that the numerical radius is not submultiplicative,

$$w(A^m) \leq w(A)^m \quad \text{for all positive integers } m.$$

For a normal matrix $A \in M_n$, we have $w(A) = \|A\|$. Thus, for any $B \in M_n$,

$$w(AB) \leq \|AB\| \leq \|A\|\|B\| = w(A)\|B\| \leq 2w(A)w(B),$$

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and also
\[ w(BA) \leq \|BA\| \leq \|B\|\|A\| = \|B\|w(A) \leq 2w(B)w(A). \]
In case \(A, B \in M_n\) are normal matrices,
\[ w(AB) \leq \|AB\| \leq \|A\|\|B\| = w(A)w(B). \]
Also, for any pairs of commuting matrices \(A, B \in M_n\),
\[ w(AB) \leq 2w(A)w(B). \]
To see this, we may assume \(w(A) = w(B) = 1\), and observe that
\[
4w(AB) = w((A + B)^2 - (A - B)^2) \leq w((A + B)^2) + w((A - B)^2)
\leq w(A + B)^2 + w(A - B)^2 \leq 8.
\]
The constant 2 is best (smallest) possible for matrices of order at least 4 because \(w(AB) = 2w(A)w(B)\) if \(A = E_{12} + E_{34}\) and \(B = E_{13} + E_{24}\), where \(E_{ij} \in M_n\) has 1 at the \((i, j)\) position and 0 elsewhere; see [2, Theorem 3.1]. In connection to this, there has been interest in studying the best (smallest) constant \(\xi > 0\) such that
\[ w(AB) \leq \xi w(A)w(B) \]
for all commuting matrices \(A, B \in M_n\) with \(n \leq 3\). For \(n = 2\), the best constant \(\xi\) is one; the existing proofs of the \(2 \times 2\) case depend on deep theory on analytic functions, von Neumann inequality, and functional calculus on operators with numerical radius equal to one, etc.; for example, see [5, 6].
Researchers have been trying to find an elementary proof for this result in view of the fact that the numerical range of \(A \in M_2\) is well understood, namely, \(W(A)\) is an elliptical disk with the eigenvalues \(\lambda_1, \lambda_2\) as foci and the length of minor axis \(\sqrt{\text{tr} A^*A - |\lambda_1|^2 - |\lambda_2|^2}\); for example, see [8, 9] and [7, Theorem 1.3.6]. The purpose of this note is to provide such a proof. Our analysis is based on elementary theory in convex analysis, co-ordinate geometry, and inequalities. Using our approach, we readily give a characterization of commuting pairs of matrices \(A, B \in M_2\) satisfying \(w(AB) = w(A)w(B)\), which was done in [2, Theorem 4.1] using yet another deep result of Ando [1] that a matrix \(A\) has numerical radius bounded by one if and only if \(A = (I - Z)^{1/2}C(A + Z)^{1/2}\) for some contractions \(C\) and \(Z\), where \(Z = Z^*\). Here is our main result.

**Theorem 1** Let \(A, B \in M_2\) be nonzero matrices such that \(AB = BA\). Then \(w(AB) \leq w(A)w(B)\). The equality holds if and only if one of the following holds.

(a) \(A\) or \(B\) is a scalar matrix, i.e. of the form \(\mu I_2\) for some \(\mu \in \mathbb{C}\).

(b) There is a unitary \(U\) such that \(U^*AU = \text{diag}(a_1, a_2)\) and \(U^*BU = \text{diag}(b_1, b_2)\) with \(|a_1| \geq |a_2|\) and \(|b_1| \geq |b_2|\).

One can associate the conditions (a) and (b) in the theorem with the geometry of the numerical range of \(A\) and \(B\) as follows. Condition (a) means that \(W(A)\) or \(W(B)\) is a single point; condition (b) means that \(W(A), W(B), W(AB)\) are line segments with three sets of end points, \(\{a_1, a_2\}, \{b_1, b_2\}, \{a_1b_1, a_2b_2\}\), respectively, such that \(|a_1| \geq |a_2|\) and \(|b_1| \geq |b_2|\).
2 Proof of Theorem 1

Let $A, B \in M_2$ be commuting matrices. We may replace $(A, B)$ by $(A/w(A), B/w(B))$ and assume that $w(A) = w(B) = 1$. We need to show that $w(AB) \leq 1$.

Since $AB = BA$, we may assume that $A = \begin{pmatrix} a_1 & a_3 \\ 0 & a_2 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix}$ and $w(A) = w(B) = 1$. The result is clear if $A$ or $B$ is normal. So, we assume that $a_3, b_3 \neq 0$. Furthermore, comparing the $(1, 2)$ entries on both sides of $AB = BA$, we see that $\frac{a_1 - a_2}{a_3} = \frac{b_1 - b_2}{b_3}$. Applying a diagonal unitary similarity to both $A$ and $B$, we may further assume that $s = \frac{(a_1 - a_2)}{a_3} \geq 0$. Let $r = \frac{1}{\sqrt{s^2 + 1}}$. We have $0 < r \leq 1$. Then $A = z_1 I + s_1 C$ and $B = z_2 I + s_2 C$ with

$$z_1 = \frac{a_1 + a_2}{2}, \quad z_2 = \frac{b_1 + b_2}{2}, \quad s_1 = \frac{a_3}{2r}, \quad s_2 = \frac{b_3}{2r}, \quad \text{and} \quad C = \begin{pmatrix} \sqrt{1 - r^2} & 2r \\ 0 & -\sqrt{1 - r^2} \end{pmatrix}.$$

Note that $W(C)$ is the elliptical disk with boundary

$$\{ \cos \theta + ir \sin \theta : \theta \in [0, 2\pi] \};$$

see [8] and [7, Theorem 1.3.6]. Replacing $(A, B)$ with $(e^{it_1} A, e^{it_2} B)$ for suitable $t_1, t_2 \in [0, 2\pi]$, if necessary, we may assume that $\Re z_1, \Re z_2 \geq 0$ and $s_1, s_2$ are real. Then $W(A)$ has boundary

$$\{ \alpha_1 + |s_1| \cos \theta + i|\alpha_2 + |s_1|r \sin \theta| : \theta \in [0, 2\pi] \}.$$

Suppose $z_1 = \alpha_1 + i \alpha_2$ with $\alpha_1 \geq 0$ and $W(A)$ touches the unit circle at the point $\cos \phi_1 + i \sin \phi_1$ with $\phi_1 \in [-\pi/2, \pi/2]$. We will show that $A$ is a convex combination of $A_0 = e^{i \phi_1} I$ and another matrix $A_1$ of the form $A_1 = i(1 - r^2) \sin \phi_1 I + \xi C$ for some $\xi \in \mathbb{R}$ such that $w(A_1) \leq 1$. To this end, we first determine $\theta_1 \in [-\pi/2, \pi/2]$ satisfying

$$\cos \phi_1 + i \sin \phi_1 = (\alpha_1 + |s_1| \cos \theta_1) + i(\alpha_2 + |s_1|r \sin \theta_1).$$

Note that the direction of the tangent at the intersection point using the parametric equation

$$x + iy = (\alpha_1 + |s_1| \cos \theta) + i(\alpha_2 + |s_1|r \sin \theta)$$

is $-\sin \theta_1 + ir \cos \theta_1$, which agrees with the direction of the tangent line of the unit circle at the point $\cos \phi_1 + i \sin \phi_1$, which is $-\sin \phi_1 + i \cos \phi_1$. As a result, we have

$$(\cos \theta_1, \sin \theta_1) = \frac{(\cos \phi_1, r \sin \phi_1)}{\sqrt{\cos^2 \phi_1 + r^2 \sin^2 \phi_1}}.$$

Furthermore, since $\cos \phi_1 + i \sin \phi_1 = -(\alpha_2 + |s_1|r \sin \theta_1) + i(\alpha_1 + |s_1| \cos \theta_1)$, we have

$$\alpha_1 = \cos \phi_1 - \frac{|s_1| \cos \phi_1}{\sqrt{\cos^2 \phi_1 + r^2 \sin^2 \phi_1}} \geq 0 \quad \text{and} \quad \alpha_2 = \sin \phi_1 - \frac{|s_1| r^2 \sin \phi_1}{\sqrt{\cos^2 \phi_1 + r^2 \sin^2 \phi_1}}.$$
Let \( \hat{s}_1 = \sqrt{\cos^2 \phi_1 + r^2 \sin^2 \phi_1} \). We show that \( |s_1| \leq \hat{s}_1 \) as follows.

If \( \cos \phi_1 > 0 \), then \( \alpha_1 = \cos \phi_1 \left( 1 - \frac{|s_1|}{\hat{s}_1} \right) \geq 0 \), and hence \( |s_1| \leq \hat{s}_1 \).

If \( \cos \phi_1 = 0 \), then \( \sin \phi_1 = \pm 1 \), \( \hat{s}_1 = r \) and \((\alpha_1, \alpha_2) = (0, \sin \phi_1(1 - |s_1|r))\) so that the parametric equation of the boundary of \( W(A) \) in (1) becomes

\[
x + iy = |s_1| \cos \theta + i(\sin \phi_1(1 - |s_1|r) + |s_1|r \sin \theta).
\]

Since \( w(A) = 1 \), for all \( \theta \in [0, 2\pi] \), we have

\[
0 \leq 1 - \left[ (|s_1| \cos \theta)^2 + (\sin \phi_1(1 - |s_1|r) + |s_1|r \sin \theta)^2 \right] = (1 - r^2)|s_1|^2(1 - \sin \phi_1 \sin \theta)^2 - 2|s_1|(|s_1| - r)(1 - \sin \phi_1 \sin \theta).
\]

As \( \sin \phi_1 = \pm 1 \), we may let \( \sin \theta = d \sin \phi_1 \) with \( d \in [0, 1] \) and deduce that

\[
0 \leq (1 - r^2)|s_1|^2(1 - d)^2 - 2|s_1|(|s_1| - r)(1 - d) \quad \text{for all} \quad d \in [0, 1].
\]

Therefore, \( |s_1| - r \leq 0 \), which gives \( |s_1| \leq r = \hat{s}_1 \).

Let

\[
A_0 = e^{i\phi_1}I \quad \text{and} \quad A_1 = i(1 - r^2) \sin \phi_1 I + \nu_1 \hat{s}_1 C,
\]

where \( \nu_1 = 1 \) if \( s_1 \geq 0 \) and \( \nu_1 = -1 \) if \( s_1 < 0 \). Then \( W(A_1) \) is the elliptical disk with boundary \( \{ \hat{s}_1 \cos \theta + i[(1 - r^2) \sin \phi_1 + \hat{s}_1 r \sin \theta] : \theta \in [0, 2\pi] \} \), and for every \( \theta \in [0, 2\pi] \),

\[
(\hat{s}_1 \cos \theta)^2 + ((1 - r^2) \sin \phi_1 + \hat{s}_1 r \sin \theta)^2 = 1 - (1 - r^2)(\hat{s}_1 \sin \theta - r \sin \phi_1)^2 \leq 1.
\]

Therefore, \( w(A_1) \leq 1 \), and \( A = \left( 1 - \frac{|s_1|}{\hat{s}_1} \right) A_0 + \frac{|s_1|}{\hat{s}_1} A_1 \) is a convex combination of \( A_0 \) and \( A_1 \).

Similarly, we can assume that \( W(B) \) touches the unit circle at \( e^{i\phi_2} \), with \( \phi_2 \in [-\pi/2, \pi/2] \). Then we can write \( B \) as a convex combination of

\[
B_0 = e^{i\phi_2}I \quad \text{and} \quad B_1 = i(1 - r^2) \sin \phi_2 I + \nu_2 \hat{s}_2 C
\]

with \( \hat{s}_2 = \sqrt{\cos^2 \phi_2 + r^2 \sin^2 \phi_2} \) and \( \nu_2 \in \{1, -1\} \). Let \( U = \begin{pmatrix} r & \sqrt{1 - r^2} \\ \sqrt{1 - r^2} & r \end{pmatrix} \). Then \( U^*CU = -C \). If \( \nu_2 = -1 \), we may replace \( (A, B) \) by \( (U^*AU, U^*BU) \) so that \( (\nu_1, \nu_2) \) will change to \( (-\nu_1, -\nu_2) \). So, we may further assume that \( r_2 = 1 \).

By the above analysis, \( AB \) is a convex combination of \( A_0B_0, A_0B_1, A_1B_0 \) and \( A_1B_1 \). Since \( w(e^{it}T) = w(T) \) for all \( t \in \mathbb{R} \) and \( T \in M_n \), the first three matrices have numerical radius 1. We will prove that

\[
w(A_1B_1) < 1.
\]

It will then follow that \( w(AB) \leq 1 \), where the equality holds only when \( A = A_0 \) or \( B = B_0 \).

For notation simplicity, let \( w_1 = \sin \phi_1 \) and \( w_2 = \sin \phi_2 \). Replacing \( (A, B) \) by \( (B, A) \), if necessary, we may assume that \( |w_1| \geq |w_2| \). If \( -w_1 \geq |w_2| \), replace \( (A, B) \) by \( (A^*, B^*) \). So, we can further assume that

\[
0 \leq |w_2| \leq w_1 \leq 1.
\]
Recall from (2) and (3) that $A_1 = i(1 - r^2)w_1 I + \nu_1 s_1 C$ and $B_1 = i(1 - r^2)w_2 I + s_2 C$ because $\nu_2 = 1$. Since $C^2 = (1 - r^2)I_2$, we have

$$A_1 B_1 = (1 - r^2)(u I_2 + iv C),$$

where

$$u = \nu_1 s_1 s_2 - w_1 w_2 (1 - r^2) \quad \text{and} \quad v = w_1 s_2 + \nu_1 w_2 s_1.$$

If $r = 1$, then $A_1 B_1 = 0$. Assume that $0 < r < 1$. We need to show that

$$\frac{1}{1 - r^2} w(A_1 B_1) = w(u I + iv C) < \frac{1}{(1 - r^2)}.$$

Because $W(u I + iv C)$ is an elliptical disk with boundary $\{u + iv(\cos \theta + ir \sin \theta) : \theta \in [0, 2\pi]\}$, it suffices to show that

$$f(\theta) = |u + iv(\cos \theta + ir \sin \theta)|^2 < \frac{1}{(1 - r^2)^2} \quad \text{for all} \quad \theta \in [0, 2\pi].$$

Note that

$$f(\theta) = (u - rv \sin \theta)^2 + (v \cos \theta)^2
= u^2 - 2r u v \sin \theta + r^2 v^2 \sin^2 \theta + v^2 (1 - \sin^2 \theta)
= \frac{u^2}{1 - r^2} + v^2 - (\sqrt{1 - r^2} v \sin \theta + \frac{ru}{\sqrt{1 - r^2}})^2
\leq \frac{u^2}{1 - r^2} + v^2
= \frac{1}{(1 - r^2)} \left( (\nu_1 \sqrt{1 - (1 - r^2)w_1^2} \sqrt{1 - (1 - r^2)w_2^2} - w_1 w_2 (1 - r^2))^2 + (1 - r^2)(w_1 \sqrt{1 - (1 - r^2)w_1^2} + \nu_1 w_2 \sqrt{1 - (1 - r^2)w_1^2})^2 \right)
= \frac{(1 - (1 - r^2)(w_1^2 - w_2^2))}{(1 - r^2)} \quad \text{because} \quad \nu_1^2 = 1
\leq \frac{1}{(1 - r^2)} \quad \text{because} \quad w_1^2 \geq w_2^2
< \frac{1}{(1 - r^2)^2} \quad \text{because} \quad 0 < r < 1.$$

Consequently, we have $w(A_1 B_1) < 1$ as asserted in (4). Moreover, by the comment after (4), if $w(AB) = w(A)w(B)$, then $A = A_0$ or $B = B_0$. Conversely, if $A = A_0$ or $B_0$, then we clearly have $W(AB) = w(A)w(B)$. The proof of the theorem is complete. \hfill \Box

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