A new criterion and a special class of $k$-positive maps

Dedicated to Professor Leiba Rodman.

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Abstract

We study $k$-positive maps on operators. We obtain new criteria on $k$-positivity in terms of the $k$-numerical range, and use it to improve and refine some earlier results on $k$-positive maps related to the study of quantum information science. We also consider a special class of positive maps extending the construction of Choi on positive maps that are not completely positive. Some open questions on the decomposability of such maps are answered.

2000 Mathematics Subject Classification. 15A69, 15A60, 15B57, 46N50.

Keywords. $k$-positive maps, $k$-numerical range.
1 Introduction

Denote by $B(H,K)$ the set of bounded linear operators from the Hilbert space $H$ to the Hilbert space $K$, and write $B(H,K) = B(H)$ if $H = K$. Let $B(H)^+ \subseteq B(K)^+$ be the set of positive semidefinite operators in $B(H)$. If $H$ and $K$ have dimensions $n$ and $m$ respectively, we identify $B(H,K)$ with the set $M_{m,n}$ of $m \times n$ matrices, and write $M_{n,n} = M_n$, and $B(H)^+ = M_n^+$.

A linear map $L : B(H) \rightarrow B(K)$ is positive if $L(B(H)^+) \subseteq B(K)^+$. For a positive integer $k$, the map $L$ is $k$-positive if the map $I_k \otimes L : M_k(B(H)) \rightarrow M_k(B(K))$ is positive, where $(I_k \otimes L)(A) = (L(A_{ij}))$ for $A = (A_{ij})_{1 \leq i,j \leq k}$ with $A_{ij} \in B(H)$. A map is completely positive if it is $k$-positive for every positive integer $k$. The study of positive maps has been the central theme for many pure and applied topics; for example, see [5, 17, 20, 23, 24, 31]. In particular, the study has attracted a lot of attention of physicists working in quantum information science in recent decades, because positive linear maps can be used to distinguish entanglement of quantum states[14]. There is considerable interest in finding positive maps that are not completely positive, which can be applied to detect entangled states (for example, see [1, 3, 4, 7, 8, 9, 10, 12, 16, 18, 19, 21, 26, 27, 28] and the references therein). Completely positive linear maps have been studied extensively by researchers. However, the structure of positive linear maps is still unclear even for the finite dimensional case [6, 11, 17, 25].

In this paper, we obtain a new criterion of $k$-positivity in terms of the $k$-numerical range. The result is used to refine a result of Chruściński and Kossakowski[7]. We also consider a new class of positive maps extending the construction of Choi on positive maps that are not completely positive. An open question on the decomposability of such maps is answered.

The paper is organized as follows. Section 2 summarizes some basic known criteria for $k$-positive maps. In Section 3, we obtain a new criterion for $k$-positivity for elementary operators in terms of the $k$-numerical range. The new criterion is used to refine the results of Chruściński and Kossakowski[7]. In Section 3, we discuss a family of positive maps, called $D$-type positive maps. They can be viewed as an extension of the construction of Choi on positive maps that are not completely positive. We give a necessary and sufficient condition for such maps to be $k$-positive. We also consider the decomposability of such maps and answer an open problem of the first and fourth authors.

In our discussion, we will use the physicists notation. The adjoint of an operator $A$ is denoted by $A^\dagger$. A vector in $H$ will be denoted by $|x\rangle$ and $\langle x|$ is the dual vector of the vector $|x\rangle$ in the dual space of $H$. The inner product of two vectors $|x\rangle$ and $|y\rangle$ in $H$ will be denoted by $\langle y|x\rangle$. If $|x\rangle \in H$ and $|y\rangle \in K$, then the rank one operator $|x\rangle \langle y| : K \rightarrow H$ will send a vector $|z\rangle \in K$ to $|x\rangle (\langle y|z\rangle) = (\langle y|z\rangle|x\rangle)$.

2 Preliminaries

In this section, we review several equivalent conditions of $k$-positivity that will be used in the subsequent discussion. Some of these conditions are known; for example, see [5, 12, 21, 30, 31].

Proposition 2.1 Suppose $L : B(H) \rightarrow B(K)$ is a linear map continuous under strong operator topology. The following are equivalent.

(a) $L$ is $k$-positive, i.e., $I_k \otimes L$ is positive.

(b) $(I_k \otimes L)(P)$ is positive semi-definite for all rank one orthogonal projection $P \in M_k(B(H))$. 

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(c) for all (orthonormal) subset \( X = \{ |x_1>, \ldots, |x_k> \} \subseteq H \), the operator matrix defined by \( L_X = (L(|x_i><x_j|))_{1 \leq i,j \leq k} \) is positive semi-definite.

Let \( \{ E_{11}, E_{12}, \ldots, E_{nn} \} \) be the standard basis for \( M_n \). Suppose \( L : M_n \to B(K) \) is a linear map. The Choi matrix \( C(L) \) is the operator matrix with \( (L(E_{ij}))_{1 \leq i,j \leq n} \). Clearly, there is a one-one correspondence between a linear map \( L \) and the Choi matrix \( C(L) \). One can use the Choi matrix to determine whether the map \( L \) is \( k \)-positive.

**Proposition 2.2** Let \( L : M_n \to B(K) \) and \( 1 \leq k \leq n \). The following are equivalent.

(a) \( L \) is \( k \)-positive.

(b) \( \langle x|C(L)|x \rangle \geq 0 \) for all \( |x> = \sum_{p=1}^{k} |y_p> \otimes |z_p> \) with \( |y_p> \otimes |z_p> \in \mathbb{C}^n \otimes K \).

(c) \( (I_n \otimes P)C(L)(I_n \otimes P) \) is positive semi-definite for all rank-\( k \) orthogonal projection \( P \in B(K) \).

A linear map \( L : B(H) \to B(K) \) is called an elementary operator[17] if it has the form

\[
L(X) = \sum_{j=1}^{k} A_j X B_j^\dagger
\]

for some \( A_1, \ldots, A_k, B_1, \ldots, B_k \in B(H, K) \). If \( H \) and \( K \) are finite dimensional, then every linear map is elementary. Since we are interested in positive linear map, we focus on linear maps which map self-adjoint operators to self-adjoint operators. It is not hard to show that every elementary operator sending the set of self-adjoint operators into itself must have the form

\[
L(X) = \sum_{j=1}^{p} C_j X C_j^\dagger - \sum_{j=1}^{q} D_j X D_j^\dagger.
\]

Hou [17] gave a condition for an elementary operator in the above form to be \( k \)-positive. One readily extends Proposition 2.1 to the following.

**Proposition 2.3** Suppose \( L : B(H) \to B(K) \) has the form (1) with \( C_1, \ldots, C_p, D_1, \ldots, D_q \in B(H, K) \). Then the following are equivalent.

(a) \( L \) is \( k \)-positive, i.e., \( I_k \otimes L \) is positive.

(b) \( \sum_{r=1}^{p} (I_k \otimes C_r)P(I_k \otimes C_r^\dagger) - \sum_{s=1}^{q} (I_k \otimes D_s)P(I_k \otimes D_s^\dagger) \) \( \in M_k(B(K))^+ \) for all rank one orthogonal projection \( P \in M_k(B(H)) \).

(c) for all (orthonormal) subset \( \{ |x_1>, \ldots, |x_k> \} \subseteq H \), \( \sum_{i,j=1}^{k} E_{ij} \otimes L(|x_i><x_j|) \) is positive semi-definite, equivalently,

\[
\sum_{r=1}^{p} \sum_{i,j=1}^{k} E_{ij} \otimes C_r |x_i><x_j| C_r^\dagger \geq \sum_{s=1}^{q} \sum_{i,j=1}^{k} E_{ij} \otimes D_s |x_i><x_j| D_s^\dagger.
\]

(d) for all \( |x> \in \mathbb{C}^k \otimes H \), there is an \( q \times p \) matrix \( T_x \) with operator norm \( \| T_x \| \leq 1 \) such that

\[
\begin{pmatrix}
I_k \otimes D_1 \\
I_k \otimes D_2 \\
\vdots \\
I_k \otimes D_q
\end{pmatrix}
\begin{pmatrix}
|x>
\end{pmatrix} = (T_x \otimes I_K)
\begin{pmatrix}
I_k \otimes C_1 \\
I_k \otimes C_2 \\
\vdots \\
I_k \otimes C_p
\end{pmatrix}
\begin{pmatrix}
|x>
\end{pmatrix}.
\]
Proof. The equivalence of (b) and (c) follows from Proposition 2.1 and the special form of \( L \).
For the equivalence of (a), (b) and (d), see [17]. □

3 A new criterion and refinements of some previous results

Recall that, for a linear operator \( A \in B(H) \) and a positive integer \( k \leq \dim H \), the \( k \)-numerical range of \( A \) is defined by

\[
W_k(A) = \left\{ \sum_{j=1}^{k} \langle x_j | A | x_j \rangle : \{ |x_1 \rangle, \ldots, |x_k \rangle \} \text{ is an orthonormal set in } H \right\}.
\]

Denote by

\[
\Gamma_k(H) = \left\{ |x\rangle = \begin{pmatrix} |x_1\rangle \\ \vdots \\ |x_k\rangle \end{pmatrix} : \{ |x_1 \rangle, \ldots, |x_k \rangle \} \text{ is an orthonormal set in } H \right\}.
\]

Also for simplicity, denote \( \Gamma_{n,k} = \Gamma_k(C^n) \). Then the \( k \)-numerical range of \( A \) can also be reformulated as

\[
W_k(A) = \{ \langle x | (I_k \otimes A) | x \rangle : |x\rangle \in \Gamma_k(H) \}.
\]

If \( \dim H = n < \infty \), and \( A \) is Hermitian with eigenvalues \( a_1 \geq \cdots \geq a_n \), then

\[
W_k(A) = \left[ \sum_{j=1}^{k} a_{n-j+1}, \sum_{j=1}^{k} a_j \right].
\]

For the details of \( k \)-numerical ranges, see [2]. The following proposition gives the relation between \( k \)-numerical ranges and \( k \)-positivity of elementary operators.

**Theorem 3.1** Suppose \( L : M_n \to B(K) \) has the form

\[
X \mapsto \sum_{r=1}^{p} C_r X C_r^\dagger - \sum_{s=1}^{q} D_s X D_s^\dagger
\]

with \( C_1, \ldots, C_p, D_1, \ldots, D_q \in B(H,K) \).

(a) If \( L \) is \( k \)-positive, then \( W_k \left( \sum_{r=1}^{p} C_r^\dagger C_r - \sum_{s=1}^{q} D_s^\dagger D_s \right) \subseteq [0, \infty) \).

(b) If for all unit vectors \( |u\rangle = (u_1, \ldots, u_p)^t \in C^p \) and \( |v\rangle = (v_1, \ldots, v_q)^t \in C^q \),

\[
\min W_k \left( \sum_{r} u_r C_r \right) \geq \max W_k \left( \sum_{s} v_s D_s \right),
\]

then \( L \) is \( k \)-positive.

Here, \( \min S \) and \( \max S \) denote the minimum and maximum value of a subset \( S \) of real numbers.
Proof. If $L$ is $k$-positive, then $(I_k \otimes L)(|x\rangle\langle x|)$ is positive semi-definite for every $|x\rangle \in \Gamma$, where $\Gamma$ is defined in (2). Then part (a) is trivially followed by taking trace and the definition of $k$-numerical range.

For (b), suppose (4) holds for all unit vectors $|u\rangle = (u_1, \ldots, u_p)^t \in \mathbb{C}^p$ and $|v\rangle = (v_1, \ldots, v_q)^t \in \mathbb{C}^q$. For $|x\rangle \in \Gamma$, let

$$\tilde{C}_x = \begin{pmatrix} C_1|x_1\rangle & \cdots & C_p|x_1\rangle \\ \vdots & \ddots & \vdots \\ C_1|x_k\rangle & \cdots & C_p|x_k\rangle \end{pmatrix} \quad \text{and} \quad \tilde{D}_x = \begin{pmatrix} D_1|x_1\rangle & \cdots & D_q|x_1\rangle \\ \vdots & \ddots & \vdots \\ D_1|x_k\rangle & \cdots & D_q|x_k\rangle \end{pmatrix}.$$ 

We will show that $\tilde{C}_x \tilde{C}_x^\dagger - \tilde{D}_x \tilde{D}_x^\dagger$ is positive semi-definite, or equivalently, for all unit vector $|y\rangle \in K^{\otimes k}$,

$$\|\langle y|\tilde{C}_x\rangle\|^2 \geq \|\langle y|\tilde{D}_x\rangle\|^2.$$ 

Denote by $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_{\min(m,n)}(A)$ the singular values of $A \in M_{m,n}$. Note that there is $|\tilde{x}\rangle \in \Gamma$ so that $\tilde{C}_x$ has the smallest $p$-th singular value $\sigma_p(\tilde{C}_x)$ among all choices of $|x\rangle \in \Gamma$. Then

$$\|\langle y|\tilde{C}_x\rangle\| \geq \sigma_p(\tilde{C}_x) \geq \sigma_p(\tilde{C}_\tilde{x}) \quad \text{for all unit vector} \quad |y\rangle \in K^{\otimes k}.$$ 

Moreover, there is a unit vector $|\tilde{u}\rangle = (\tilde{u}_1, \ldots, \tilde{u}_p)^t \in \mathbb{C}^p$ such that

$$(\sigma(\tilde{C}_\tilde{x}))^2 = \|\tilde{C}_\tilde{x}|\tilde{u}\rangle\|^2 = \left\|\left(\sum_{r=1}^k \tilde{u}_r C_r \tilde{x}_1\right) \cdots \left(\sum_{r=1}^k \tilde{u}_r C_r \tilde{x}_k\right)\right\|^2$$

$$= \sum_{j=1}^k \langle \tilde{x}_j | \left(\sum_{r=1}^k \tilde{u}_r C_r \right)^* \left(\sum_{r=1}^k \tilde{u}_r C_r \right) | \tilde{x}_j \rangle$$

$$\geq \min W_k \left( \sum_{r=1}^k |\tilde{u}_r C_r|^2 \right).$$

Similarly, we can choose $|\tilde{x}\rangle \in \Gamma$ so that $\tilde{D}_\tilde{x}$ has the largest maximum singular value $\sigma_1(\tilde{D}_\tilde{x})$ among all choice of $|x\rangle \in \Gamma$. Then

$$\|\langle y|\tilde{D}_x\rangle\| \leq \sigma_1(\tilde{D}_\tilde{x}) \leq \hat{\sigma}_1(\tilde{D}_\tilde{x}).$$

Moreover, there is a unit vector $|\tilde{v}\rangle = (\tilde{v}_1, \ldots, \tilde{v}_q)^t \in \mathbb{C}^q$ such that

$$\max W_k \left( \sum_{s} v_s D_s \right)^* \left( \sum_{s} v_s D_s \right) \geq \sum_{j=1}^k \langle \tilde{x}_j | \left(\sum_{s} \tilde{v}_s D_s \right)^* \left(\sum_{s} \tilde{v}_s D_s \right) | \tilde{x}_j \rangle$$

$$\geq \left\|\left(\sum_{s} \tilde{v}_s D_s \right)^* \left(\sum_{s} \tilde{v}_s D_s \right)\right\|^2$$

$$\|\tilde{D}_\tilde{x}|\tilde{v}\rangle\|^2 = (\sigma(\tilde{D}_\tilde{x}))^2.$$ 

By our assumption, we have $\sigma_p(\tilde{C}_\tilde{x}) \geq \sigma_1(\tilde{D}_\tilde{x})$, and hence

$$\|\langle y|\tilde{C}_x\rangle\| \geq \sigma_p(\tilde{C}_\tilde{x}) \geq \sigma_1(\tilde{D}_\tilde{x}) \geq \|\langle y|\tilde{D}_x\rangle\|.$$ 

The desired conclusion follows. \qed
Remark  Note that in the above proof, if there is \( |x\rangle \in \Gamma_{n,k} \) such that \( \tilde{C}_x = 0 \), then

\[
\min \left\{ W_k \left( \sum_r u_r C_r \right) \dagger \left( \sum_r u_r C_r \right) : |u\rangle = (u_1, \ldots, u_p)^t \in \mathbb{C}^p, \langle u | u \rangle = 1 \right\} = 0.
\]

On the other hand, if

\[
\min \left\{ W_k \left( \sum_r u_r C_r \right) \dagger \left( \sum_r u_r C_r \right) : |u\rangle = (u_1, \ldots, u_p)^t \in \mathbb{C}^p, \langle u | u \rangle = 1 \right\} > 0,
\]

then \( \tilde{C}_x \) has rank \( kp \) for all \( |x\rangle \in \Gamma_{n,k} \).

In the following, we give a numerical range criterion for \( k \)-positivity of maps \( \phi : M_n \rightarrow M_m \) defined by

\[
L(X) = L_1(X) - L_2(X)
\]

with

\[
L_1(X) = \sum_{j=1}^{p} \gamma_j F_j X F_j^\dagger \quad \text{and} \quad L_2(X) = \sum_{j=p+1}^{mn} \gamma_j F_j X F_j^\dagger,
\]

\( \gamma_1, \ldots, \gamma_{mn} \geq 0 \),

where \( F_1, \ldots, F_{mn} \in M_{m,n} \) satisfy the following:

(C1) \( \{ F_j \}_{j=1}^{mn} \subseteq M_{m,n} \) is an orthonormal set using the inner product \( \langle X, Y \rangle = \text{tr} (X Y^\dagger) \).

The positivity of such maps were also considered in [7, 8] using the norm

\[
\|X\|_k = \left\{ \sum_{j=1}^{k} \sigma_j (X)^2 \right\}^{1/2},
\]

where \( \sigma_1 (X) \geq \sigma_2 (X) \geq \cdots \geq \sigma_{\min (m,n)} (X) \) are the singular values of \( X \in M_{m,n} \). The norm \( \|X\|_k \) is known as the \((2,k)\)-spectral norm [22]. The authors of [7] mistakenly referred this as the Ky Fan \( k \)-norm \( |X|_k = \sum_{j=1}^{k} \sigma_j (X) \) of the matrix \( X \); see [13, p.445].

Consider the following condition on \( \{ F_j \}_{j=1}^{mn} \subseteq M_{mn} \).

(C2) for all orthonormal basis \( \{|x_1\rangle, |x_2\rangle, \ldots, |x_n\rangle\} \) of \( \mathbb{C}^n \),

\[
\left\{ P_k = \sum_{i,j=1}^{n} |x_i\rangle \langle x_j| \otimes F_k |x_i\rangle \langle x_j| F_k^\dagger : 1 \leq k \leq mn \right\}
\]

is a set of mutually orthogonal rank one matrices.

Notice that \( P_k \) is indeed the Choi matrix of the map \( A \mapsto F_k A F_k^\dagger \). In [7], the authors showed that under the assumption (C2), if \( 1 > \sum_{j=p+1}^{mn} \| F_j \|_k^2 \) and

\[
\sum_{i,j=1}^{n} E_{ij} \otimes L_1 (E_{ij}) \geq \frac{\sum_{j=p+1}^{mn} \gamma_j \| F_j \|_k^2}{1 - \sum_{j=p+1}^{mn} \| F_j \|_k^2} \left( I_n \otimes I_m - \sum_{j=p+1}^{mn} P_j \right),
\]

then \( \phi \) is \( k \)-positive.

In [8], the authors stated the result using the assumption (C1) instead of (C2) without explaining their relations. In addition, there are some typos in the papers that further obscured the
results. In fact, it can be deduced from [31, Proposition 4.14] (the special case for \(m = n\) was also treated in [29, Lemma 1]) that the conditions \((C1)\) and \((C2)\) are equivalent. In the following, we use results in the previous sections to refine and improve the results in \([7, 8]\) using the concept of the \(k\)-numerical range.

**Proposition 3.2** Suppose that \(\{F_j : 1 \leq j \leq mn\}\) is an orthonormal basis of \(M_{m,n}\) and \(L : M_n \rightarrow M_m\) has the form

\[
L(X) = \sum_{j=1}^{p} \gamma_j F_j X F_j^\dagger - \sum_{j=p+1}^{mn} \gamma_j F_j X F_j^\dagger, \quad \gamma_1, \ldots, \gamma_{mn} \geq 0.
\]

Assume that \(1 \leq k \leq \min\{m, n\}\) and \(\xi_k = 1 - \max W_k(\sum_{j=p+1}^{mn} F_j F_j^\dagger) > 0\).

(a) If

\[
\gamma_i \geq \xi_k^{-1} \max W_k \left( \sum_{j=p+1}^{mn} \gamma_j F_j F_j^\dagger \right), \quad i = 1, \ldots, p,
\]

then \(L\) is \(k\)-positive.

(b) If \(p = mn - 1\) and

\[
\gamma_i < \xi_k^{-1} \gamma_{mn} \max W_k \left( F_{mn} F_{mn}^\dagger \right) = \xi_k^{-1} \gamma_{mn} ||F_{mn}||^2_k \quad \text{for all} \quad i = 1, \ldots, mn - 1,
\]

then \(L\) is not \(k\)-positive.

**Proof.** (a) Let \(w_k = \max W_k \left( \sum_{j=p+1}^{mn} \gamma_j F_j F_j^\dagger \right)\). Suppose \(\gamma_i \geq \xi_k^{-1} w_k\) for each \(i = 1, \ldots, p\). We show that \(I_k \otimes L(\langle x | x \rangle)\) is positive semidefinite for all \(\langle x | \rangle \in \Gamma_{n,k}\), where \(\Gamma_{n,k}\) is defined in (2). The conclusion will then follow from Proposition 2.3.

We may extend \(\langle x | \rangle \in \Gamma_{n,k}\) to \(\langle \bar{x} | \rangle \in \Gamma_{n,n}\) with \(\bar{x} = \left( \begin{array}{c} |x_1 \rangle \\ \vdots \\ |x_n \rangle \end{array} \right)\) such that \(\{ |x_1 \rangle, \ldots, |x_n \rangle \}\) is an orthonormal basis for \(C^n\). Since the conditions \((C1)\) and \((C2)\) are equivalent, the matrices \(\{P_1, \ldots, P_{mn}\}\) defined in \((C2)\) form a set of mutually orthogonal set of rank one matrices. Thus, \(\sum_{r=1}^{mn} P_r = I_{mn}\) and hence

\[
\sum_{r=1}^{mn} (F_r |x_i \rangle \langle x_j | F_r^\dagger)_{1 \leq i, j \leq n} = I_{mn}.
\]

Focusing on the leading \(mk \times mk\) principal submatrix, we have

\[
\sum_{r=1}^{p} (F_r |x_i \rangle \langle x_j | F_r^\dagger)_{1 \leq i, j \leq k} = I_{mk} - \sum_{r=p+1}^{mn} (F_r |x_i \rangle \langle x_j | F_r^\dagger)_{1 \leq i, j \leq k}. \quad (5)
\]

Note that

\[
\text{tr} \left( \sum_{r=p+1}^{mn} \gamma_r (F_r |x_i \rangle \langle x_j | F_r^\dagger)_{1 \leq i, j \leq k} \right) = \sum_{j=1}^{k} \langle x_j | \sum_{r=p+1}^{mn} \gamma_r F_r^\dagger F_r | x_j \rangle \leq w_k.
\]
Thus,
\[ \sum_{r=p+1}^{mn} \gamma_r (F_r^\dagger x_i) (x_j | F_r^\dagger)_{1 \leq i, j \leq k} \leq w_k I_{mk}. \tag{6} \]

Applying this argument to the special case when \( \gamma_{p+1} = \cdots = \gamma_{mn} \), we see that
\[ \sum_{r=p+1}^{mn} (F_r^\dagger x_i) (x_j | F_r^\dagger)_{1 \leq i, j \leq k} \leq \max_{j=p+1}^{mn} F_j^* F_j \]

By (5) and the fact that \( \xi_k = 1 - \max_{j=p+1}^{mn} F_j^* F_j \), we have
\[ \xi_k I_{mk} \leq \sum_{r=1}^{p} (F_r^\dagger x_i) (x_j | F_r^\dagger)_{1 \leq i, j \leq k}. \]

Because \( w_k \xi_k^{-1} \leq \gamma_i \) for each \( i = 1, \ldots, p \), we have
\[ w_k I_{mk} \leq w_k \xi_k^{-1} \sum_{r=1}^{p} (F_r^\dagger x_i) (x_j | F_r^\dagger)_{1 \leq i, j \leq k} \leq \sum_{r=1}^{p} \gamma_r (F_r^\dagger x_i) (x_j | F_r^\dagger)_{1 \leq i, j \leq k}. \tag{7} \]

By (6) and (7), we have the desired operator inequality
\[ \sum_{r=p+1}^{mn} \gamma_r (F_r^\dagger x_i) (x_j | F_r^\dagger)_{1 \leq i, j \leq k} \leq \sum_{r=1}^{p} \gamma_r (F_r^\dagger x_i) (x_j | F_r^\dagger)_{1 \leq i, j \leq k}. \]

(b) Suppose that the hypothesis of (b) holds. We can choose \( |x_1\rangle, \ldots, |x_k\rangle \) in \( \mathbb{C}^n \) so that
\[ \text{tr} \ (F_{mn}|x_i\rangle \langle x_j| F_{mn}^\dagger)_{1 \leq i, j \leq k} = \max_{j=p+1}^{mn} (F_{mn}^\dagger F_{mn}) = \| F_{mn} \|^2_k, \]
i.e., the rank one matrix \( \gamma_{mn} (F_{mn}|x_i\rangle \langle x_j| F_{mn}^\dagger)_{1 \leq i, j \leq k} \) has a nonzero eigenvalue \( \gamma_{mn} \| F_{mn} \|^2_k \). Now,
\[ \sum_{r=1}^{mn-1} \gamma_r (F_r^\dagger x_i) (x_j | F_r^\dagger)_{1 \leq i, j \leq k} < \xi_k^{-1} \gamma_{mn} \left( \sum_{r=1}^{mn-1} (F_r^\dagger x_i) (x_j | F_r^\dagger)_{1 \leq i, j \leq k} \right) \leq \gamma_{mn} I_{mk}. \]

Thus, the matrix
\[ \sum_{r=1}^{mn-1} \gamma_r (F_r^\dagger x_i) (x_j | F_r^\dagger)_{1 \leq i, j \leq k} - \gamma_{mn} (F_{mn}|x_i\rangle \langle x_j| F_{mn}^\dagger)_{1 \leq i, j \leq k} \]
has a negative eigenvalue. The result follows from Proposition 2.3. \( \square \)

Part (b) of the Proposition 3.2 was proved in [7, 8]. Using the fact that
\[ \max_{j=p+1}^{mn} \sum_{r=p+1}^{mn} \gamma_r F_r^\dagger F_r \leq \sum_{r=p+1}^{mn} \gamma_r \| F_r \|^2_k \]
for all nonnegative numbers \( \gamma_{p+1}, \ldots, \gamma_{mn} \), we can deduce the main result in [7, 8].

8
Corollary 3.3 Suppose \( \{F_j : 1 \leq j \leq mn\} \) is an orthonormal basis of \( M_{m,n} \) and a map \( L : M_n \to M_m \) has the form

\[
L(X) = \sum_{j=1}^{p} \gamma_j F_j X F_j^\dagger - \sum_{j=p+1}^{mn} \gamma_j F_j X F_j^\dagger, \quad \gamma_1, \ldots, \gamma_{mn} \geq 0.
\]

Assume that \( 1 \leq k \leq \min\{m, n\} \) and \( \tilde{\xi}_k = 1 - \sum_{j=p+1}^{mn} \|F_j\|_k^2 > 0 \). If

\[
\gamma_i \geq \tilde{\xi}_k^{-1} \left( \sum_{j=p+1}^{mn} \gamma_j \|F_j\|_k^2 \right) \quad \text{for all} \quad i = 1, \ldots, p,
\]

then \( L \) is \( k \)-positive.

In [8], the authors mistakenly claimed that if \( \sum_{j=p+1}^{mn} \|F_j\|_{k+1}^2 < 1 \) and

\[
\gamma_i < \tilde{\xi}_{k+1}^{-1} \left( \sum_{j=p+1}^{mn} \gamma_j \|F_j\|_{k+1}^2 \right) \quad \text{for all} \quad i = 1, \ldots, p,
\]

then \( L \) is not \((k + 1)\)-positive, see [8, Theorem 1]. The case when \( p = mn - 1 \) was proved in [7, Theorem 1] and also Proposition 3.2. However, the following example shows that the claim in general does not hold when \( p < mn - 1 \). Also, this example demonstrates that Proposition 3.2 is stronger than Corollary 3.3 (the result in [7, 8]).

Example 3.4 Let \( m = n = 8 \). Suppose \( \{F_1, \ldots, F_{64}\} \) is an orthonormal basis for \( M_8 \) such that

\[
F_{63} = \frac{1}{4} I_4 \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad F_{64} = \frac{1}{4} I_4 \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\]

Define \( L : M_8 \to M_8 \) by

\[
L(X) = \sum_{j=1}^{62} \gamma_j F_j X F_j^\dagger - \sum_{j=63}^{64} F_j X F_j^\dagger.
\]

Then \( \|F_{63}\|_2^2 + |F_{64}|_2^2 = 1/2 + 1/2 = 1 \) and \( W_2 \left( \sum_{j=63}^{64} F_j^\dagger F_j \right) = W_2(I_8/4) = \{1/2\} \). So, Corollary 3.3 is not applicable. By Proposition 3.2, the map \( L \) is 2-positive if \( \gamma_i \geq 1 \) for \( i = 1, \ldots, 62 \).

4 \quad \textbf{D-type linear maps}

In this section, we consider linear maps \( L : M_n \to M_n \) of the form

\[
A = (a_{ij}) \mapsto \text{diag} \left( \sum_{k=1}^{n} a_{kk} d_{k1}, \ldots, \sum_{k=1}^{n} a_{kk} d_{kn} \right) - A
\]  

for an \( n \times n \) nonnegative matrix \( D = (d_{ij}) \). This type of maps will be called D-type linear maps. The question of when a D-type map is positive was studied intensively by many authors and applied in quantum information theory to detect entangled states and construct entanglement witnesses [18, 26]. For example, if \( D = (n - 1)I_n + E_{12} + \cdots + E_{n-1,n} + E_{n,1} \), we get a positive map

\[
A = (a_{ij}) \mapsto (n - 1) \text{diag} (a_{11}, a_{22}, a_{33}, \ldots, a_{nn}) + \text{diag} (a_{nn}, a_{11}, a_{22}, \ldots, a_{n-1,n-1}) - A,
\]

which is not completely positive. This can be viewed as a generalization of the Choi map [5].
4.1 Criteria for $k$-positivity

For $\gamma \geq 0$, define $L_\gamma : M_n \to M_n$ by

$$L_\gamma(A) = \gamma (\text{tr } A) I_n - A.$$  

This is a well-known example in literature. It is known that for all $k \in \{1, \ldots, n\}$, $L_\gamma$ is $k$-positive if and only if $\gamma \geq k$, for example, see [31, p.119], which can also be proved easily by Proposition 2.1. Notice that the map $L_\gamma$ is a $D$-type map with all entries of $D$ being $\gamma$.

In the following, we present a necessary and sufficient criteria of $D$-type linear map to be $k$-positive.

**Proposition 4.1** Suppose $L : M_n \to M_n$ is a $D$-type map (8) for an $n \times n$ nonnegative matrix $D = (d_{ij})$. The following conditions are equivalent.

(a) $L$ is $k$-positive.

(b) $d_{jj} > 0$ for all $j = 1, \ldots, n$, and for all $k \times n$ matrix $U$ with columns $|u_1\rangle, \ldots, |u_n\rangle \in \mathbb{C}^k$ satisfying $\text{tr } (U^\dagger U) = 1$, we have

$$\sum_{j=1}^{n} |u_j\rangle \langle u_j| (U \text{diag } (d_{1j}, \ldots, d_{nj}) U^\dagger)|[-1]|u_j\rangle \leq 1,$$

where $X^{-1}$ is the Moore-Penrose generalized inverse of $X$.

**Proof.** for all $k \times n$ matrix $U$ with column $|u_1\rangle, \ldots, |u_n\rangle \in \mathbb{C}^k$ satisfying $\text{tr } (U^\dagger U) = 1$, consider the unit vector

$$|u\rangle = \sum_{j=1}^{n} |u_j\rangle \otimes |\hat{e}_{j}\rangle \in \mathbb{C}^{nk},$$

which can also be expressed as

$$|u\rangle = \sum_{j=1}^{k} |e_j\rangle \otimes |\hat{u}_j\rangle,$$

where $\{|e_1\rangle, \ldots, |e_k\rangle\}$ and $\{|\hat{e}_1\rangle, \ldots, |\hat{e}_n\rangle\}$ are the standard basis of $\mathbb{C}^k$ and $\mathbb{C}^n$, respectively, and $|\hat{u}_j\rangle$ is the $j$-th column of $U^T$. Let

$$F_{ij} = L(|\hat{u}_i\rangle \langle \hat{u}_j|) + |\hat{u}_i\rangle \langle \hat{u}_j| = \sum_{\ell=1}^{n} (|\hat{u}_j\rangle \langle \text{diag } (d_{1\ell}, \ldots, d_{n\ell}) |\hat{u}_i\rangle) |\hat{e}_\ell\rangle \langle \hat{e}_\ell|,$$

which is a diagonal matrix with the $\ell$-th diagonal entry equal to $\langle \hat{u}_j | \text{diag } (d_{1\ell}, \ldots, d_{n\ell}) |\hat{u}_i\rangle$. By Proposition 2.1, $L$ is $k$-positive if and only if the $nk \times nk$ matrix

$$(I_k \otimes L)(|u\rangle \langle u|) = \sum_{i,j=1}^{k} |e_i\rangle \langle e_j| \otimes (F_{ij} - |\hat{u}_i\rangle \langle \hat{u}_j|) = \left( \sum_{i,j=1}^{k} |e_i\rangle \langle e_j| \otimes F_{ij} \right) - |u\rangle \langle u|$$

is positive semi-definite. Notice that the above matrix is permutationally similar to

$$\sum_{i,j=1}^{k} (F_{ij} - |\hat{u}_i\rangle \langle \hat{u}_j|) \otimes |e_i\rangle \langle e_j| = \left( \sum_{i,j=1}^{k} F_{ij} \otimes |e_i\rangle \langle e_j| \right) - |\hat{u}\rangle \langle \hat{u}|,$$

is positive semi-definite. Notice that the above matrix is permutationally similar to

$$\sum_{i,j=1}^{k} (F_{ij} - |\hat{u}_i\rangle \langle \hat{u}_j|) \otimes |e_i\rangle \langle e_j| = \left( \sum_{i,j=1}^{k} F_{ij} \otimes |e_i\rangle \langle e_j| \right) - |\hat{u}\rangle \langle \hat{u}|,$$
where $|\hat{u}\rangle = \sum_{j=1}^{n}|\hat{e}_j\rangle \otimes |u_j\rangle$. Recall that $|\hat{u}_j\rangle$ is the $j$-th column of $U^T$. Direct computation shows that
\[
\sum_{i,j=1}^{k} F_{ij} \otimes |e_i\rangle\langle e_j| = \sum_{i,j=1}^{k} \sum_{\ell=1}^{n} ((\hat{u}_j) |\text{diag} (d_{1\ell}, \ldots, d_{n\ell})|\hat{u}_i\rangle) |\hat{e}_\ell\rangle \otimes |e_i\rangle\langle e_j|
\]
\[
= \sum_{\ell=1}^{n} |\hat{e}_\ell\rangle \langle \hat{e}_\ell| \otimes \left( \sum_{i,j=1}^{k} ((\hat{u}_j) |\text{diag} (d_{1\ell}, \ldots, d_{n\ell})|\hat{u}_i\rangle) |e_i\rangle\langle e_j| \right)^T
\]
\[
= \sum_{\ell=1}^{n} |\hat{e}_\ell\rangle \langle \hat{e}_\ell| \otimes \left( (U^T)^{*} \text{diag} (d_{1\ell}, \ldots, d_{n\ell}) U^{\dagger} \right)
\]
\[
= \sum_{\ell=1}^{n} |\hat{e}_\ell\rangle \langle \hat{e}_\ell| \otimes \left( U \text{diag} (d_{1\ell}, \ldots, d_{n\ell}) U^{\dagger} \right)
\]
\[
= D_1 \oplus \cdots \oplus D_n,
\]
where $D_\ell = U \text{diag} (d_{1\ell}, \ldots, d_{n\ell}) U^{\dagger}$ for $\ell = 1, \ldots, n$. Therefore, the condition is equivalent to:

(c) $|\hat{u}\rangle$ lies in the range of $D_1 \oplus \cdots \oplus D_n$ and $\|D_1 \oplus \cdots \oplus D_n\|^{1/2} \leq 1$. Note that for all choice of $U$ satisfying $\text{tr} (U^{\dagger} U) = 1$ the corresponding $|\hat{u}_j\rangle$ always lies in the range of $D_j$ for $j = 1, \ldots, n$ if and only if $d_{jj} > 0$ for all $j = 1, \ldots, n$; the norm inequality in (c) is the same as the inequality stated in (b). Therefore, condition (a) is equivalent to condition (c), which is equivalent to condition (b). □

Notice that the inequality in Proposition 4.1(b) is equivalent to
\[
\text{tr} \left( (U \text{diag} (d_{1j}, \ldots, d_{nj}) U^{\dagger})^{[-1]} U U^{\dagger} \right) \leq 1.
\]
Also, Proposition 4.1 is particularly useful when $k = 1$.

**Corollary 4.2** Let $L : M_n \to M_n$ be a $D$-type map of the form (8) with $D = (d_{ij})$. For $u = (u_1, u_2, \ldots, u_n)^t \in \mathbb{C}^n$, let $f_j(u) = \sum_{i=1}^{n} d_{ij} |u_i|^2$. Then, $L$ is positive if and only if any one of the following equivalent conditions hold

1. $d_{ii} > 0$ for all $i = 1, \ldots, n$ and $\sum_{j \neq 0} |u_j|^2 f_j(u) \leq 1$ for every unit vector $|u\rangle = (u_1, u_2, \ldots, u_n)^t \in \mathbb{C}^n$.

2. $d_{ii} > 0$ for all $i = 1, \ldots, n$ and $\sum_{j=1}^{n} |u_j|^2 f_j(u) \leq 1$ for every vector $|u\rangle = (u_1, u_2, \ldots, u_n)^t \in \mathbb{C}^n$ with $u_i \neq 0$ for all $i = 1, \ldots, n$.

**Proof.** For $k = 1$, (1) is equivalent to condition (b) in Proposition 4.1. (2) is equivalent to (1) because $|u_j|^2 f_j(u)$ is continuous and homogeneous in $u$. □

### 4.2 Construction of $D$-type positive maps

In this section, we discuss how to construct $D$-type positive linear maps using the results in previous sections. Recall that a permutation $\pi$ of $(i_1, \ldots, i_\ell)$ is an $\ell$-cycle if $\pi(i_j) = i_{j+1}$ for $j = 1, \ldots, \ell - 1$ and $\pi(i_\ell) = i_1$. Note that every permutation $\pi$ of $(1, \ldots, n)$ has a disjoint cycle decomposition $\pi = (\pi_1)(\pi_2) \cdots (\pi_r)$, that is, there exists a set $\{F_s\}_{s=1}^{r}$ of disjoint cycles of $\pi$ with $\cup_{s=1}^{r} F_s = \{1, 2, \ldots, n\}$ such that $\pi_s = |F_s|$ and $\pi(i) = \pi_s(i)$ whenever $i \in F_s$. We have the following.
Proposition 4.3 Suppose \( \pi \) is a permutation of \((1, 2, \ldots, n)\) with disjoint cycle decomposition \((\pi_1) \cdots (\pi_r)\) such that the maximum length of \(\pi_i\) is equal to \(\ell > 1\) and \(P_\pi = (\delta_{\pi(i)j})\) is the permutation matrix associated with \(\pi\). For \(t \geq 0\), let \(\Phi_{t, \pi} : M_n \rightarrow M_n\) be the \(D\)-type map of the form (8) with \(D = (n-t)I_n + tP_\pi\). Then \(\Phi_{t, \pi}\) is positive if and only if \(t \leq \frac{n}{\ell}\).

Proof. It is easily checked that for \(0 \leq t \leq 1\), the function
\[
g(r_1, r_2, \ldots, r_s) = \sum_{i=1}^{s} \frac{1}{s - t + tr_i} \leq 1 \quad \text{for all } r_i > 0 \text{ and } r_1r_2 \cdots r_s = 1,\tag{9}
\]
and the function \(g\) attains the maximum 1 when \(r_1 = \cdots = r_s = 1\).

Suppose \(0 \leq t \leq \frac{n}{\ell}\). We are going to use condition (2) in Corollary 4.2 to show that \(\Phi_{t, \pi}\) is positive. For all vector \(|u\rangle = (u_1, u_2, \ldots, u_n)^t \in C^n\), with \(u_i \neq 0\) for all \(i = 1, \ldots, n\), we have \(f_i(u) = (n-t)|u_i|^2 + t|u_{\pi(i)}|^2\). So, by Corollary 4.2, \(\Phi_{t, \pi}\) is positive if
\[
f(u_1, u_2, \ldots, u_n) = \sum_{i=1}^{n} \frac{|u_i|^2}{(n-t)|u_i|^2 + t|u_{\pi(i)}|^2} \leq 1\tag{10}
\]
for all vector \(|u\rangle = (u_1, u_2, \ldots, u_n)^t\) with nonzero entries.

Suppose \(\pi\) is a product of \(r\) disjoint cycles, that is, \(\pi = (\pi_1)(\pi_2) \cdots (\pi_r)\). Let \(F_j\) be the set of indices corresponding to the cycle \(\pi_j\) and \(\ell_j\) denote the number of elements in \(F_j\) for \(j = 1, \ldots, r\). Then \(\ell = \max\{\ell_1, \ldots, \ell_r\}\) and \(\sum_j \ell_j = n\). For all vector \(|u\rangle = (u_1, u_2, \ldots, u_n)^t \in C^n\), with \(u_i \neq 0\) for all \(i = 1, \ldots, n\), we have \(\prod_{i \in F_j} \frac{|u_{\pi(i)}|^2}{|u_i|^2} = 1\). It follows that
\[
f(u_1, u_2, \ldots, u_n) = \frac{n}{(n-t)|u_i|^2 + t|u_{\pi(i)}|^2} = \sum_{j=1}^{r} \sum_{i \in F_j} \frac{|u_i|^2}{(n-t)|u_i|^2 + t|u_{\pi(i)}|^2} \leq \sum_{j=1}^{r} \frac{\ell_j}{n} \cdot 1 = 1,
\]
whenever \(0 \leq \frac{\ell_j}{n} t \leq 1\) for all \(1 \leq j \leq r\) by (9), or equivalently, \(0 \leq t \leq \frac{n}{\ell_j} \leq \frac{n}{\ell}\). Therefore, (10) holds.

Conversely, suppose \(t > \frac{n}{\ell}\). Let \(\pi = (\pi_1)(\pi_2) \cdots (\pi_r)\) be a decomposition of \(\pi\) into disjoint cycles. Without loss of generality, we may assume that \(\ell_1 = \ell \geq \ell_j\) for all \(j = 2, \ldots, r\), and \(\pi_1\) is a cycle on \((1, 2, \ldots, \ell)\). Let \(u_i = e^{\frac{i}{\ell}}, 0 < \epsilon < 1 - \frac{n}{\ell}\) for \(i = 1, \ldots, \ell\) and \(u_i = 1\) for \(\ell + 1 \leq i \leq n\). Then we have
\[
f(u_1, u_2, \ldots, u_n) = \sum_{i=1}^{\ell-1} \frac{1}{(n-t) + \ell} + \frac{1}{(n-t) + \ell} + \sum_{i=\ell+1}^{n} \frac{1}{(n-t) + t} \geq \frac{\ell - 1}{(n-t) + t} + \frac{n - \ell}{n} \geq \frac{\ell - 1}{(n-t) + t (\frac{1 - n}{\ell})} + \frac{n - \ell}{n} = \frac{\ell - 1}{n - \frac{n}{\ell}} + \frac{n - \ell}{n} = \frac{\ell + n - \ell}{n} = 1,
\]
which implies $\Phi_{t,\pi}$ is not positive.

Next, we consider a general map $\Lambda_D$ of the form (8).

**Proposition 4.4** Let $\Lambda_D : M_n \to M_n$ have the form (8) for a nonnegative matrix $D = (d_{ij})$ with all row sum and column sum equal to $n$. Then $\Lambda_D$ is positive if $d_{ii} \geq (n - 1)$ for all $i = 1, \ldots, n$. Moreover, the following conditions are equivalent.

(a) $\Lambda_D$ is completely positive. (b) $\Lambda_D$ is 2-positive. (c) $D = nI_n$.

**Proof.** Suppose $d_{ii} \geq n - 1$ for all $i = 1, \ldots, n$. Then $D = (n - 1)I_n + S$ for a doubly stochastic matrix $S$, which is a convex combination of permutation matrices (for example, see [13, Theorem 8.7.1, pp. 527]). We may represent $S$ as

$$S = \sum_{i=1}^{m} p_i P_{\pi_i}$$

for some permutations $\pi_1, \pi_2, \ldots, \pi_m$ of $\{1, 2, \ldots, n\}$ and positive scalars $p_i$ with $\sum_{i=1}^{m} p_i = 1$. Let $S_i = (n - 1)I_n + P_{\pi_i}$ and $\Lambda_{S_i}$ be the linear map of the form as in (8). By Proposition 4.3, $\Lambda_{S_i}$ is a positive map. Thus, $\Lambda_D$ is a convex combination of positive maps, and is therefore positive.

Next, we prove the three equivalent conditions. The implication. (a) $\Rightarrow$ (b) is clear. For (c) $\Rightarrow$ (a), it is well known and easy to check, say, by considering the Choi matrix, that $\Lambda_D$ is completely positive if $D = nI_n$.

It remains to prove (b) $\Rightarrow$ (c). Suppose $D \neq nI_n$. Then $d_{ii} < n$ for some $i$. Without loss of generality, we assume that $i = 1$. Let

$$U = (|u_1\rangle \cdots |u_n\rangle) = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \end{pmatrix} \in M_{2,n}$$

and define

$$D_j := U \text{diag}(d_{1j}, d_{2j}, \ldots, d_{nj}) U^\dagger = \frac{1}{n} \begin{pmatrix} d_{1j} & 0 \\ 0 & n - d_{1j} \end{pmatrix}, \quad j = 1, \ldots, n.$$ 

As $n > d_{11}$,

$$\langle \hat{u}_1 | D_j^{-1} | \hat{u}_1 \rangle = \frac{1}{d_{11}} > \frac{1}{n} \quad \text{and} \quad \langle \hat{u}_j | D_j^{-1} | \hat{u}_j \rangle = \frac{1}{n - d_{1j}} \geq \frac{1}{n} \quad \text{for } j = 2, \ldots, n.$$ 

Hence,

$$\sum_{j=1}^{n} \langle \hat{u}_j | D_j^{-1} | \hat{u}_j \rangle > 1,$$

and $\Lambda_D$ is not 2-positive by Proposition 4.1. □

In [26], the positive map $\Lambda_D$ with $D = (n - 1)I_n + P$ for a permutation matrix $P$ was considered, and the special case when $P$ is a length $n$-cycle was discussed in details. By Propositions 4.3 and 4.4, we have the following corollary.

**Corollary 4.5** Let $\Lambda_D : M_n \to M_n$ be a $D$-type map of the form (8) with $D = (n - 1)I_n + P$ for a permutation matrix $P$. Then $\Lambda_D$ is positive. Moreover, the following are equivalent.

(a) $\Lambda_D$ is completely positive. (b) $\Lambda_D$ is 2-positive. (c) $D = nI_n$. 13
The condition $d_{ii} \geq n - 1$ for each $i$ is not necessary for $\Lambda_D$ in Proposition 4.4 to be positive as seen below.

**Example 4.6** Let $D = \begin{pmatrix} 1.35 & 1 & 0.65 \\ 0.65 & 1.35 & 1 \\ 1 & 0.65 & 1.35 \end{pmatrix}$. Here, $d_{ii} < 2 = 3 - 1$. Direct computation shows that $\sum_{j=1}^{3} \frac{|u_{ij}|^2}{f_j(a)} \leq 1$ for all $(u_1, u_2, u_3) \in \mathbb{C}^3$. Therefore, $\Lambda_D$ is positive by Corollary 4.2.

**Example 4.7** In Proposition 4.3, let $0 \leq t \leq 1$ and $D = (d_{ij}) = (n - t)I_n + tS$, where $S = \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_2 & s_3 & \cdots & s_1 \end{pmatrix}$ with $s_i \geq 0$ ($i = 1, 2, \ldots, n$) and $\sum_{i=1}^{n} s_i = 1$. Define $\Lambda_D : M_n \to M_n$ by

$$\Lambda_D((a_{ij})) = \begin{pmatrix} f_1 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & f_2 & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & f_n \end{pmatrix},$$

where

$$f_1 = (n - t - 1 + ts_1)a_{11} + ts_n a_{22} + ts_{n-1}a_{33} + \cdots + ts_2 a_{nn},$$
$$f_2 = ts_2 a_{11} + (n - t - 1 + ts_1)a_{22} + ts_n a_{33} + \cdots + ts_3 a_{nn},$$
$$\vdots$$
$$f_n = ts_n a_{11} + ts_{n-1}a_{22} + ts_{n-2}a_{33} + \cdots + (n - t - 1 + ts_1)a_{nn}. $$

By Proposition 4.3, the map $\Lambda_D$ is positive.

Finally, we give an example which illustrates how to apply Proposition 4.3 to construct positive elementary operators for all dimension.

**Example 4.8** Let $H$ and $K$ be Hilbert spaces of dimension at least $n$, and let $\{|e_i\rangle\}_{i=1}^{n}$ and $\{\hat{e}_j\rangle\}_{j=1}^{n}$ be any orthonormal sets of $H$ and $K$, respectively. For all permutation $\pi \neq \text{id}$ of $\{1, 2, \ldots, n\}$, let $l(\pi) = l \leq n$. Let $\Phi_{t, \pi} : \mathcal{B}(H) \to \mathcal{B}(K)$ be defined by

$$\Phi_{t, \pi}(A) = (n - t) \sum_{i=1}^{n} E_{i \pi(i)} A E_{i \pi(i)}^\dagger + t \sum_{i=1}^{n} E_{i \pi(i)} A E_{i \pi(i)}^\dagger - (\sum_{i=1}^{n} E_{i \pi(i)}) A (\sum_{i=1}^{n} E_{i \pi(i)})^\dagger \text{ for all } A \in \mathcal{B}(H),$$

where $E_{j i} = |\hat{e}_j\rangle \langle e_i|$. Then $\Phi_{t, \pi}$ is positive if and only if $0 \leq t \leq \frac{n}{l}$.

In fact, for the case $\dim H = \dim K = n$, $\Phi_{t, \pi}$ is a $D$-type map of the form (8) with $D = (n - t)I + tP_\pi$ as discussed in Proposition 4.3.
4.3 Decomposability of $D$-type positive maps

Decomposability of positive linear maps is a topic of particular importance in quantum information theory since it is related to the PPT states (that is, the states with positive partial transpose). In this section, we will give a new class of decomposable positive linear maps.

The following result is well known (see [15]).

**Proposition 4.9** Suppose $L : M_n \to M_m$ has the form (8). Then $L$ is decomposable if and only if the Choi matrix $C(L)$ is a sum of two matrices $C_1$ and $C_2$ such that $C_1$ and the partial transpose of $C_2$ are positive semi-definite.

In [26], it was shown that the linear maps $\Phi^{(k)} = \Phi_{1,\pi}$ with $\pi(i) = i + k \pmod{n}$ in Proposition 4.3 are indecomposable whenever either $n$ is odd or $k \neq \frac{n}{2}$. It was asked in [26] that whether or not $\Phi^{(\frac{n}{2})}$ is decomposable when $n$ is even. In this section, we will answer this question by showing that $\Phi^{(\frac{n}{2})}$ is decomposable. In fact, this is a special case of the following proposition as $(\pi)^2 = id$.

**Proposition 4.10** Let $\pi$ be a permutation of $\{1, 2, \ldots, n\}$. If $\pi^2 = id$, then the positive linear map $\Phi_{1,\pi}$ in Proposition 4.3 is decomposable.

**Proof.** For simplicity, denote $\Phi = \Phi_{1,\pi}$. Let $F$ be the set of fixed points of $\pi$. Since $\Phi(E_{ii}) = (n-2)E_{ii} + E_{\pi(i),\pi(i)}$ and $\Phi(E_{ij}) = -E_{ij}$, the Choi matrix of $\Phi$ is

$$C(\Phi) = \sum_{i=1}^{n} (n-2)E_{ii} \otimes E_{ii} + \sum_{i=1}^{n} E_{\pi(i),\pi(i)} \otimes E_{ii} - \sum_{i \neq j} E_{ij} \otimes E_{ij}$$

$$= \sum_{i \in F} (n-1)E_{ii} \otimes E_{ii} + \sum_{i \notin F} (n-2)E_{ii} \otimes E_{ii}$$

$$- \sum_{i \neq j, \pi(i) \neq j} E_{ij} \otimes E_{ij} + \sum_{i \notin F} E_{\pi(i),\pi(i)} \otimes E_{ii} - \sum_{i \neq j, \pi(i) \neq j} E_{i,\pi(i)} \otimes E_{i,\pi(i)}.$$

Let

$$C_1 = \sum_{i \in F} (n-1)E_{ii} \otimes E_{ii} + \sum_{i \notin F} (n-2)E_{ii} \otimes E_{ii} - \sum_{i \neq j, \pi(i) \neq j} E_{ij} \otimes E_{ij}$$

and

$$C_2 = \sum_{i \notin F} E_{\pi(i),\pi(i)} \otimes E_{ii} - \sum_{i \notin F} E_{i,\pi(i)} \otimes E_{i,\pi(i)}.$$  

Since $\pi^2 = id$ the cardinal number of $F^c$ must be even. Thus we have

$$C_2 = \sum_{i < \pi(i)} (E_{\pi(i),\pi(i)} \otimes E_{ii} + E_{ii} \otimes E_{\pi(i),\pi(i)} - E_{i,\pi(i)} \otimes E_{i,\pi(i)} - E_{\pi(i),i} \otimes E_{\pi(i),i}).$$

As

$$C_2^{T_2} = \sum_{i < \pi(i)} (E_{\pi(i),\pi(i)} \otimes E_{ii} + E_{ii} \otimes E_{\pi(i),\pi(i)} - E_{i,\pi(i)} \otimes E_{i,\pi(i)} - E_{\pi(i),i} \otimes E_{\pi(i),i}) \geq 0,$$

we see that $C_2$ is PPT.

Observe that $C_1 \cong A \oplus 0$, where $A = (a_{ij}) \in M_n$ is a Hermitian matrix satisfying $a_{ii} = n-2$ or $n-1$, $a_{ij} = 0$ or $-1$ so that $\sum_{j=1}^{n} a_{ij} = 0$. It is easily seen from the strictly diagonal dominance theorem [13, Theorem 6.1.10, pp. 349] that $A$ is positive semi-definite. So $C_1 \geq 0$, and by Proposition 4.9, $\Phi$ is decomposable. \qed
Acknowledgments

Research of Hou was supported by the NNSF of China (11171249) and a grant from International Cooperation Program in Sciences and Technology of Shanxi (2011081039). Research of Li was supported by the 2011 Shanxi 100 Talent program, a USA NSF grant, and a HK RGC grant. He is an honorary professor of University of Hong Kong and Shanghai University. Research of Poon was supported by a USA NSF grant and a HK RGC grant. Research of Qi was supported by the NNSF of China (11101250) and Youth Foundation of Shanxi Province (2012021004). Research of Sze was supported by a HK RGC grant PolyU 502411.

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