NUMERICAL RANGE, DILATION, AND COMPLETELY POSITIVE MAPS

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This paper is dedicated to Professor Tsuyoshi Ando.

Abstract. A proof using the theory of completely positive maps is given to the fact that if $A \in M_2$ or $A \in M_3$ has a reducing eigenvalue, then every bounded linear operator $B$ with $W(B) \subseteq W(A)$ has a dilation of the form $I \otimes A$. This gives a unified treatment for the different cases of the result obtained by researchers using different techniques.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators acting on a Hilbert space $\mathcal{H}$ with inner product $\langle x, y \rangle$. If $\mathcal{H}$ has dimension $n$, we identify $\mathcal{B}(\mathcal{H})$ with $M_n$ and $\mathcal{H} = \mathbb{C}^n$ with $\langle x, y \rangle = y^*x$.

The numerical range of $A \in \mathcal{B}(\mathcal{H})$ is defined and denoted by

$$W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H}, \langle x, x \rangle = 1\}.$$ 

We say that an operator $B \in \mathcal{B}(\mathcal{H})$ admits a dilation $A \in \mathcal{B}(\mathcal{K})$ if there is a partial isometry $X : \mathcal{H} \to \mathcal{K}$ such that $X^*X = I_\mathcal{H}$ and $X^*AX = B$. If $B$ admits a dilation $I_{\mathcal{K}_1} \otimes A \in \mathcal{B}(\mathcal{K}_1 \otimes \mathcal{K})$ for some Hilbert space $\mathcal{K}_1$, we will simply say that $B$ admits a dilation of the form $I \otimes A$.

There are interesting connections between the numerical range inclusion and dilation relation between two operators. For example, the following is known, see [7, 8].

**Theorem 1.1.** Let $A \in M_3$ be a normal matrix. If $B \in \mathcal{B}(\mathcal{H})$ satisfies $W(B) \subseteq W(A)$, then $B$ admits a dilation of the form $I \otimes A$.

Also, we have the following [1, 2, 5].

**Theorem 1.2.** Let $A \in M_2$. If $B \in \mathcal{B}(\mathcal{H})$ satisfies $W(B) \subseteq W(A)$, then $B$ admits a dilation of the form $I \otimes A$.

The following theorem was proved in [6] that generalizes Theorem 1.1 and 1.2.

**Theorem 1.3.** Suppose $A \in M_3$ has a non-trivial reducing subspace. If $B \in \mathcal{B}(\mathcal{H})$ satisfies $W(B) \subseteq W(A)$, then $B$ admits a dilation of the form $I \otimes A$.

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Furthermore, it was shown in [5] that the conclusion of Theorem 1.3 would fail if one considers a general matrix $A \in M_5$ or a normal matrix $A \in M_4$. The proofs in [1, 2, 5, 6, 7, 8] used different techniques. In this note, we give a unified proof of the above results. In particular, we will give a proof of the theorems in Section 1 in an equivalent form involving unital positive and completely positive maps. We first introduce some background.

An operator system $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is a self-adjoint subspace of $\mathcal{B}(\mathcal{H})$ which contains $I_\mathcal{H}$. A linear map $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{K})$ is unital if $\Phi(I_\mathcal{H}) = I_\mathcal{K}$, $\Phi$ is positive if $\Phi(A)$ is positive semi-definite for every positive semi-definite $A \in \mathcal{S}$, and $\Phi$ is said to be completely positive if $I_k \otimes \Phi : M_k(\mathcal{S}) \to M_k(\mathcal{B}(\mathcal{K}))$ defined by $(T_{ij}) \mapsto (\Phi(T_{ij}))$ is positive for every $k \geq 1$.

Suppose $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Let $\mathcal{S}$ be the operator system spanned by $\{I_\mathcal{H}, A, A^*\}$. Define a unital linear map $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{K})$ by $\Phi(aI + bA + cA^*) = aI + bB + cB^*$. By [6, Lemma 4.1], $\Phi$ is positive if and only if $W(B) \subseteq W(A)$. On the other hand, Stinespring’s representation theorem [10] (see also the paragraphs after Theorem 4.1 and Theorem 4.6 in [9]) shows that $\Phi$ is completely positive if and only if $B$ has a dilation of the form $I \otimes A$. Therefore, Theorems 1.2 – 1.3 can be stated in the following form.

**Theorem 1.4.** Suppose $A = A_0$ or $A_0 \oplus [\mu]$ with $A_0 \in M_2$ and $B \in \mathcal{B}(\mathcal{H})$. Define a linear map $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ by

$$\Phi(aI + bA + cA^*) = aI + bB + cB^*$$

for any $a, b, c \in \mathbb{C}$.

Then $\Phi$ is positive if and only if $\Phi$ is completely positive.

2. Proofs

To prove Theorem 1.4, we need several lemmas, some of which are well known. In our discussion, we will let $E_{ij}$ be the basic matrix unit of appropriate size, and write $P \geq 0$ if a matrix or operator $P$ is positive semi-definite.

**Lemma 2.1.** ([9, Corollary 6.7]) Let $\mathcal{S}$ be an operator system. Then every positive linear map $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ is completely positive for every Hilbert space $\mathcal{H}$ if and only if every positive linear map $\Psi : \mathcal{S} \to M_n$ is completely positive for all positive integer $n$.

Recall that $f : \mathbb{R}^m \to \mathbb{R}^m$ is an affine map if it has the form $x \mapsto Rx + x_0$ for a real matrix $R \in M_m$ and $x_0 \in \mathbb{R}^m$. The affine map is invertible if $R$ is invertible, and the inverse of $f$ has the form $y \mapsto R^{-1}y - R^{-1}x_0$. One can extend the definition of affine map to an $m$-tuple of self-adjoint operators in $\mathcal{B}(\mathcal{H})$ by

$$(A_1, \ldots, A_m) \mapsto (A_1, \ldots, A_m)(r_{ij}I_\mathcal{H}) + (B_0, \ldots, B_m)$$

for a real matrix $R = (r_{ij}) \in M_m$ and an $m$-tuple $(B_1, \ldots, B_m)$ of self-adjoint operators in $\mathcal{B}(\mathcal{H})$. We have the following result which can be easily verified.

**Lemma 2.2.** Let $\mathcal{S}$ be an operator system with a basis $\{I, A_1, \ldots, A_m\}$, and $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ a unital linear map defined by $\Phi(A_j) = B_j \in \mathcal{B}(\mathcal{H})$ for $j = 1, \ldots, m$. Suppose $f$ is an invertible affine map such that $f(A_1, \ldots, A_m) = (\hat{A}_1, \ldots, \hat{A}_m)$ and $f(B_1, \ldots, B_m) = (\hat{B}_1, \ldots, \hat{B}_m)$. Then $\Phi$ is positive (respectively, completely
positive) if and only if the unital map \( \hat{\Phi} \) defined by \( \hat{\Phi}(\hat{A}_j) = \hat{B}_j \) for \( j = 1, \ldots, m \), if positive (respectively, completely positive).

**Lemma 2.3.** Let \( S = \text{span}\{E_{jj} : 1 \leq j \leq m\} \subseteq M_m \). A linear map \( \Phi : S \rightarrow M_n \) is completely positive if and only if \( \Phi(E_{jj}) \geq 0 \) for \( j = 1, \ldots, m \). As a result, every positive linear map \( \Phi : S \rightarrow M_n \) is completely positive.

**Proof.** If \( \Phi : S \rightarrow M_n \) is positive, then \( \Phi(E_{jj}) \geq 0 \) for all \( j = 1, \ldots, m \).

Suppose \( \Phi(E_{jj}) \geq 0 \) for all \( j = 1, \ldots, m \). Let \( C = (C_{ij}) \in M_k(S) \) be positive semi-definite for a positive integer \( k \). Then \( C = C_{11} \otimes E_{11} + \cdots + C_{mm} \otimes E_{mm} \geq 0 \), where \( C_{jj} \geq 0 \) for \( j = 1, \ldots, m \). Thus,

\[
(I_k \otimes \Phi)(C) = C_{11} \otimes \Phi(E_{11}) + \cdots + C_{mm} \otimes \Phi(E_{mm}) \geq 0.
\]

Hence, \( \Phi \) is completely positive. \( \square \)

**Lemma 2.4.** Let \( S = \text{span}\{(E_{jj} : 1 \leq j \leq m) \cup \{E_{12} + E_{21}\}\} \) in \( M_n \) with \( m \geq 2 \). Then every positive linear map \( \Phi : S \rightarrow M_n \) is completely positive.

**Proof.** Suppose \( \Phi : S \rightarrow M_n \) is a positive map. If \( m = 2 \), the result is due to Choi [4, Theorem 7]. The proof in [4] relies on a result of Calderon [3]. We give a short and direct proof using basic theory of completely positive maps for completeness as follows.

Suppose \( \Phi \) is positive. Then \( \Phi(E_{11}), \Phi(E_{22}) \geq 0 \), and for any real number \( d \),

\[
\Phi(E_{11} + d(E_{12} + E_{21}) + d^2 E_{22}) = \Phi(E_{11}) + d\Phi(E_{12} + E_{21}) + d^2 \Phi(E_{22}) \geq 0.
\]

Let \( C = (C_{ij}) \in M_k(S) \) be positive semi-definite for a positive integer \( k \). Then

\[
C = C_{11} \otimes E_{11} + C_{22} \otimes E_{22} + C_{12} \otimes E_{12} + C_{21} \otimes E_{21}
\]

such that \( C_{21} = C_{12} = C_{21}^* \) so that \( C_{12} = C_{21} \) is Hermitian, and

\[
Q = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}
\]

is positive semi-definite. We need to show that

\[
(I_k \otimes \Phi)(C) = C_{11} \otimes \Phi(E_{11}) + C_{12} \otimes \Phi(E_{12} + E_{21}) + C_{22} \otimes \Phi(E_{22}) \geq 0.
\]

We focus on the case when \( C_{11} \) is invertible. The general case can be derived by continuity argument. We may replace \( C_{ij} = C_{11}^{-1/2}C_{ij}C_{11}^{-1/2} \) for \( i, j \in \{1, 2\} \) in (2.2), and assume that \( C_{11} = I \). Because \( C_{12} = C_{12}^* \), we may further replace \( C_{ij} \) by \( U^*C_{ij}U \) in (2.2) for \( i, j \in \{1, 2\} \), and assume that \( C_{11} = I \) and \( C_{12} = C_{21} = D = \text{diag}(d_1, \ldots, d_k) \) with \( d_1, \ldots, d_k \in \mathbb{R} \). Since \( Q \geq 0 \), we have

\[
\tilde{C}_{22} = C_{22} - C_{21}C_{11}^{-1}C_{12} = C_{22} - D^2 \geq 0.
\]

As \( \phi(E_{22}) \) and \( \tilde{C}_{22} \) are positive semi-definite, it follows from (2.1) that

\[
C_{11} \otimes \Phi(E_{11}) + C_{12} \otimes \Phi(E_{12} + E_{21}) + C_{22} \otimes \Phi(E_{22}) = I \otimes \Phi(E_{11}) + D \otimes \Phi(E_{12} + E_{21}) + D^2 \otimes \phi(E_{22}) + \tilde{C}_{22} \otimes \Phi(E_{22}) \geq 0
\]

as asserted.
Suppose $m > 2$ and $k$ is a positive integer. Let $C = (C_{ij}) \in M_k(S)$ be positive semi-definite. Then
\[
C = C_{11} \otimes E_{11} + C_{22} \otimes E_{22} + C_{12} \otimes E_{12} + C_{21} \otimes E_{21} + \sum_{j=3}^{m} C_{jj} \otimes E_{jj},
\]
where $\sum_{1 \leq r, s \leq 2}(C_{rs} \otimes E_{rs}) \geq 0$ and $C_{jj} \geq 0$ for all $3 \leq j \leq m$. We need to show that
\[
(I_k \otimes \Phi)(C) = \sum_{1 \leq r, s \leq 2} C_{rs} \otimes \Phi(E_{rs}) + \sum_{j=3}^{m} C_{jj} \otimes \Phi(E_{jj})
\]
is positive semi-definite. Since $\sum_{j=3}^{m} C_{jj} \otimes \Phi(E_{jj}) \geq 0$, it suffices to prove that
\[
\sum_{1 \leq r, s \leq 2} C_{rs} \otimes \Phi(E_{rs}) \geq 0,
\]
which is true because the restriction of $\Phi$ to $\{E_{12}, E_{21}, E_{12} + E_{21}\}$ is positive, and is completely positive by the result when $m = 2$. The asserted result follows. □

Now, we are ready to present the following.

**Proof of Theorem 1.4.** By Lemma 2.1, it suffices to show that if $\Phi : S \rightarrow M_n$ is positive then it is completely positive, where $S = \text{span}\{I, A, A^*\}$ satisfies the assumption of the theorem. We assume that $A$ is not a scalar matrix to avoid trivial consideration.

We will use the fact that the conclusion will not change if replace $A$ by $\alpha I + \beta U^* A U$ for any unitary matrix $U$, and $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$. Furthermore, a unital linear map $\Phi : S \rightarrow M_n$ is positive if and only if $B = \Phi(A)$ satisfies $W(B) \subseteq W(A)$ [6, Lemma 4.1]. Applying an affine transform to $A = A_1 + i A_2$ with $(A_1, A_2) = (A_1^*, A_2^*)$ will always mean applying a (real) affine transform to $(A_1, A_2)$.

First, suppose $A$ is normal. If $A \in M_2$ has eigenvalues $a_1, a_2$, we may assume that $A = \text{diag}(a_1, a_2)$. By Lemma 2.2, we may apply an invertible affine map to $A$ and assume that $(a_1, a_2) = (1, 0)$. Then $\hat{S} = \text{span}\{E_{11}, E_{22}\}$. By Lemma 2.3, every unital positive linear map $\Phi : \hat{S} \rightarrow M_n$ is completely positive.

Suppose $A \in M_3$ is normal. By Lemma 2.2, we can apply an invertible affine map and assume that (1) $A = \text{diag}(1, r, 0)$ with $r \in [0, 1]$ if the three eigenvalues of $A$ are collinear, or (2) $A = \text{diag}(1, i, 0)$ otherwise.

Suppose (1) holds, and $\Phi : \hat{S} \rightarrow M_n$ is a unital positive linear map with $\Phi(A) = B$. Then $\Phi$ is positive if and only if $x^* B x \in W(A) = [0, 1]$ for all unit vector $x \in \mathbb{C}^n$, i.e., $B$ is a positive semi-definite contraction. One can extend $\Phi$ to a map from $\hat{S} = \text{span}\{E_{11}, E_{22}, E_{33}\} \subseteq M_3$ to $M_n$ with $\Phi(E_{11}) = B, \Phi(E_{22}) = 0, \Phi(E_{33}) = I - B$. By Lemma 2.3, $\Phi$ is completely positive on $\hat{S}$, and therefore so must be the restriction map on $S$.

If (2) holds, then $\hat{S} = \text{span}\{E_{11}, E_{22}, E_{33}\}$. By Lemma 2.3, every unital positive linear map $\Phi : \hat{S} \rightarrow M_n$ is completely positive.

Next, we consider the case when $A$ is not normal. Suppose $A \in M_2$. We may replace $A$ by $U^* \left( e^{i} (A - (\text{tr}A)I/2) \right) U$ for a suitable unitary $U \in M_2$, and assume that $A = \begin{pmatrix} \alpha & b \\ 0 & -\alpha \end{pmatrix}$ with $\alpha \geq 0$ and $b > 0$. Then
$W(A)$ is an elliptical disk with major axis $[-r, r]$ and minor axis $i[-b, b]$ with $r = \sqrt{a^2 + b^2}$. We may further apply an affine transform

$$A = A_1 + iA_2 \mapsto \frac{1}{r}A_1 + \frac{i}{b}A_2.$$ 

Then $A$ is unitarily similar to the symmetric matrix $C = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$, where $W(C)$ is the unit disk centered at the origin. So, we may assume that $S = \text{span}\{I_2, C, C^*\} = \text{span}\{E_{11}, E_{22}, E_{12} + E_{21}\}$ is the set of symmetric matrices in $M_2$. By Lemma 2.4, every positive linear map $\Phi : \text{span}\{E_{11}, E_{22}, E_{12} + E_{21}\} \rightarrow M_n$ is completely positive.

Finally, suppose $A = A_0 \oplus [\mu]$. If $\mu \in W(A_0)$, then $W(A) = W(A_0)$ and the result follows from the previous case. So we can assume that $\mu \notin W(A_0)$. We may apply an affine transform to $A_0$ as in the preceding case so that $W(A_0)$ is the unit disk centered at the origin, and $A_0 \in M_2$ is nilpotent with norm 2. Applying the same affine transform to $A$ yields $A = A_0 \oplus [\hat{\mu}]$. Now, replacing $A$ by $e^{it}(U^*AU - \hat{\mu}I)$ for a suitable $t \in \mathbb{R}$, we may assume that $A = (rI_2 + C) \oplus [0]$, where $r = |\hat{\mu}| > 1$, and $C = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$. So, $A = \begin{pmatrix} r + i & 1 \\ 1 & r - i \end{pmatrix} \oplus [0]$ with $r > 1$.

Suppose $B = H + iG$ with $H = H^*$ and $G = G^*$. We will construct $P$ with $0 \leq P \leq I$ and a unital positive linear map $\Psi : \text{span}\{E_{11}, E_{22}, E_{33}, E_{12} + E_{21}\} \rightarrow M_n$ with

\begin{align}
\Psi(E_{11} + E_{22}) &= P, \\
\Psi(E_{12} + E_{21}) &= H - rP, \\
\Psi(E_{11} - E_{22}) &= G, \\
\Psi(E_{33}) &= I - P.
\end{align}

By Lemma 2.4, $\Psi$ is completely positive. One easily checks that $\Phi$ is the restriction of $\Psi$ on $S$, and is also unital completely positive.

Let $t_0 \in (0, \pi)$ be such that $\cos(t_0) = -1/r$. For any unit vector $x \in \mathbb{C}^n$, we have

\begin{align}
\langle Bx, x \rangle &= W(B) \subseteq W(A) \\
&= \{a + ib : |b|\sqrt{r^2 - 1} \leq a \text{ and } (a - r)\cos t + b\sin t \leq 1 \text{ for all } t \in [-t_0, t_0] \}.
\end{align}

Equivalently,

\begin{align}
\pm \sqrt{r^2 - 1}G &\leq H \\
\text{and}
\end{align}

\begin{align}
\cos t(H - rI) + \sin tG &\leq I \quad \text{for all } t \in [-t_0, t_0].
\end{align}

Now, the desired map $\Psi$ satisfying (2.3) is positive if and only if

$$\Psi\left(\begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix} \oplus [0]\right)$$

$$= \frac{1}{2} \left(\cos^2 \theta(P + G) + 2 \cos \theta \sin \theta(H - rP) + \sin^2 \theta(P - G)\right)$$
is positive semi-definite for all \( \theta \in \mathbb{R} \), equivalently,

\[
P \geq \cos t(H - rP) + \sin tG \quad \text{for all } t \in [-\pi, \pi].
\]

By (2.4), \( H \geq 0 \) and there exists a contraction \( C = C^* \in M_n \) such that \( G = \frac{1}{\sqrt{r^2 - 1}}H^{1/2}CH^{1/2} \). First, we show that for \( Q = \frac{I}{r+1} + \frac{C^2}{r^2-1} \),

\[
Q \geq \cos t(I - rQ) + \sin t\sqrt{r^2 - 1}C
\]

To see this, apply a unitary similarity to \( C \) and assume that \( C = \text{diag}(c_1, \ldots, c_n) \) with \( c_j \in [-1, 1] \). Then by Cauchy-Schwarz inequality and the fact that \( c_j^2 \in [0, 1] \), we have

\[
\cos t \left( \frac{I}{r+1} - \frac{r c_j^2}{r^2 - 1} \right) + \sin t \frac{c_j}{\sqrt{r^2 - 1}} \leq \sqrt{\left( \frac{1}{r+1} - \frac{r c_j^2}{r^2 - 1} \right)^2 + \frac{c_j^2}{r^2 - 1}}
\]

for each \( j = 1, \ldots, n \). Hence (2.7) holds, and for \( K = \frac{H}{r+1} + \frac{H^{1/2}C^2H^{1/2}}{r^2-1} \geq 0 \), we have

\[
\cos tH + \sin tG \leq (1 + r \cos t)K \quad \text{for all } t \in [-\pi, \pi].
\]

Suppose \( V \) is unitary such that \( K = V^*\text{diag}(d_1, \ldots, d_n)V \) with \( d_1 \geq \cdots \geq d_n \geq 0 \). Let

\[
P = \min\{K, I\} = V^*\text{diag}(p_1, \ldots, p_n)V \quad \text{with } p_j = \min\{k_j, 1\} \quad \text{for } j = 1, \ldots, n.
\]

Then for \( |t| \leq t_0 \), it follows from (2.5) and (2.8) that

\[
\cos tH + \sin tG \leq (1 + r \cos t) \min\{I, K\} \leq (1 + r \cos t)P.
\]

For \( \pi \geq |t| > t_0 \), we have \( 1 + r \cos t < 0 \). Together with (2.8), we also have

\[
\cos tH + \sin tG \leq (1 + r \cos t)K \leq (1 + r \cos t)P.
\]

Thus,

\[
\cos tH + \sin tG \leq (1 + r \cos t)P \quad \text{for all } t \in [-\pi, \pi].
\]

Hence, (2.6) holds, and the result follows.

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