Linear maps transforming the higher numerical ranges

Chi-Kwong Li a, Yiu-Tung Poon b,∗, Nung-Sing Sze c

aDepartment of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA
bDepartment of Mathematics, Iowa State University, Ames, IA 50011, USA
cDepartment of Mathematics, The University of Hong Kong, Hong Kong

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Abstract

Let \( k \in \{1, \ldots, n\} \). The \( k \)-numerical range of \( A \in \mathbb{M}_n \) is the set
\[
W_k(A) = \{ \frac{\tr X^* A X}{k} : X \text{ is } n \times k, \ X^* X = I_k \},
\]
and the \( k \)-numerical radius of \( A \) is the quantity
\[
w_k(A) = \max \{|z| : z \in W_k(A)\}.
\]
Suppose \( k > 1 \), \( k' \in \{1, \ldots, n'\} \) and \( n' < C(n, k) \min\{k', n' - k\} \). It is shown that there is
a linear map \( \phi : \mathbb{M}_n \to \mathbb{M}_{n'} \) satisfying \( W_{k'}(\phi(A)) = W_k(A) \) for all \( A \in \mathbb{M}_n \) if and only if
\( n'/n = k'/k \) or \( n'/n = k'/(n - k) \) is a positive integer. Moreover, if such a linear map \( \phi \)
exists, then there are unitary matrix \( U \in \mathbb{M}_{n'} \) and nonnegative integers \( p, q \) with \( p + q = n'/n \)
such that \( \phi \) has the form
\[
A \mapsto U^* \left[ \begin{array}{c}
A \\
\vdots \\
A^t \\
\vdots \\
A^t
\end{array} \right]_{p+q} U
\]
or
\[
A \mapsto U^* \left[ \begin{array}{c}
\psi(A) \\
\vdots \\
\psi(A) \\
\vdots \\
\psi(A)
\end{array} \right]_{p+q} U,
\]

∗ Corresponding author.
E-mail addresses: ckl@math.wm.edu (C.-K. Li), ytpoon@iastate.edu (Y.-T. Poon), nungsingsze@graduate.hku.hk (N.-S. Sze).

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where \( \psi : M_n \to M_n \) has the form \( A \mapsto (\operatorname{tr} A)I_n - (n - k)A/k \). Linear maps \( \tilde{\phi} : M_n \to M_n' \) satisfying \( w_k(\tilde{\phi}(A)) = w_k(A) \) for all \( A \in M_n \) are also studied. Furthermore, results are extended to triangular matrices.

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1. Introduction

There has been a great deal of interest in studying linear operator \( \phi : \mathcal{M} \to \mathcal{M} \), where \( \mathcal{M} \) is a matrix algebra or space, with a certain special property such as:

(a) \( f(\phi(A)) = f(A) \) for all \( A \in \mathcal{M} \), where \( f \) is a given function on \( \mathcal{M} \);
(b) \( \phi(\mathcal{S}) \subseteq \mathcal{F} \) or \( \phi(\mathcal{S}) = \mathcal{F} \) for a certain subset \( \mathcal{S} \subseteq \mathcal{M} \);
(c) \( \phi(A) \sim \phi(B) \) in \( \mathcal{M} \) whenever \( A \sim B \) in \( \mathcal{M} \) for a certain relation \( \sim \) on \( \mathcal{M} \).

Very often, \( \phi \) has nice forms such as

\[ A \mapsto MAN \quad \text{or} \quad A \mapsto MA'N \]

for some suitable \( M, N \in \mathcal{M} \). One may see [19] for a survey on the subject. Recently, there has been research on more general problems concerning linear transformations \( \phi : \mathcal{M} \to \mathcal{M}' \) with some special properties such as

(a) \( f'(\phi(A)) = f(A) \) for all \( A \in \mathcal{M} \), where \( f \) and \( f' \) are appropriate functions on \( \mathcal{M} \) and \( \mathcal{M}' \);
(b) \( \phi(\mathcal{S}) \subseteq \mathcal{F}' \) or \( \phi(\mathcal{S}) = \mathcal{F}' \) for certain subsets \( \mathcal{S} \subseteq \mathcal{M} \) and \( \mathcal{S}' \subseteq \mathcal{M}' \);
(c) \( \phi(A) \sim' \phi(B) \) in \( \mathcal{M}' \) whenever \( A \sim B \) in \( \mathcal{M} \) for certain relations \( \sim' \) on \( \mathcal{M} \) and \( \mathcal{M}' \).

Such problems are more challenging and their study often lead to the discovery of unexpected results and hidden structures of the matrix algebras \( \mathcal{M} \) and \( \mathcal{M}' \); see [6, 10]. In this paper, we consider these types of problems. We solve a specific problem and develop some proof techniques that may be useful for future study in this area.

Let us first introduce some notations and definitions. Denote by \( M_n \) the algebra of \( n \times n \) complex matrices. For \( 1 \leq k \leq n \), define (see Halmos [11]) the \( k \)-numerical range of \( A \in M_n \) as

\[ W_k(A) = \{ (\operatorname{tr} X^*AX)/k : X \text{ is } n \times k, \ X^*X = I_k \} \]

Since \( W_n(A) = \{ \operatorname{tr} A/n \} \), we always assume that \( k < n \) to avoid trivial consideration. When \( k = 1 \), we have the classical numerical range \( W_1(A) \), which is useful
in studying matrices and operators; see [11]. Researchers have studied linear maps 
\( \phi : M_n \to M_n \) such that
\[
W_k(\phi(A)) = W_k(A) \quad \text{for all } A \in M_n. \tag{1.1}
\]
By a result of Pellegrini [18], a linear map \( \phi : M_n \to M_n \) satisfies (1.1) for \( k = 1 \) if
and only if there is a unitary \( U \in M_n \) such that \( \phi \) has the form
\[
(S1) \quad A \mapsto U^*AU \quad \text{or} \quad A \mapsto U^*A^tU.
\]
Pierce and Watkins [20] extended the result of Pellegrini to other values of \( k \) as long
as \( k = n/2 \), and raised the open problem for the case \( k = n/2 \). In [12] (see also [17]),
it was shown that for \( k = n/2 \), a linear map \( \phi : M_n \to M_n \) satisfies (1.1) if and only
if there is a unitary \( U \in M_n \) such that \( \phi \) has the form (S1), or
\[
(S2) \quad n = 2k
\]
and
\[
A \mapsto (\text{tr} \, A/k)I_n - U^*AU \quad \text{or} \quad A \mapsto (\text{tr} \, A/k)I_n - U^*A^tU. \tag{1.2}
\]
In fact, for any \( k \in \{1, \ldots, n - 1\} \), a mapping \( \phi \) of the form (1.2) satisfies
\[
(n - k)W_{n-k}(A) = kW_k(\phi(A)) \quad \text{for all } A \in M_n.
\]
In [6] the authors studied linear maps \( \phi : M_n \to M_{n'} \) such that (1.1) holds with \( k = 1 \).
It was shown that for \( n' \leq 2n - 2 \), a linear map \( \phi : M_n \to M_{n'} \) satisfies (1.1) if
and only if \( n' \geq n \) and there exist a unitary \( U \in M_{n'} \) and a unital positive linear map \( f : M_n \to M_{n'} \) such that \( \phi \) has the form
\[
A \mapsto U^*[A \oplus f(A)]U \quad \text{or} \quad A \mapsto U^*[A^t \oplus f(A)]U.
\]
However, for \( n' > 2n - 2 \), there are other linear maps \( \phi : M_n \to M_{n'} \) satisfying (1.1)
with complicated structure. The complete characterization of \( \phi : M_n \to M_{n'} \) satisfying
(1.1) is unknown.

The purpose of this paper is to study those linear operators \( \phi : M_n \to M_{n'} \) satisfying
\[
W_k(\phi(A)) = W_k(A) \quad \text{for all } A \in M_n.
\]
By modifying the map in [6], we can easily get a map \( \phi : M_n \to M_{n'} \) with complicated
structure satisfying
\[
W_k(\phi(A)) = W_1(A) \quad \text{for all } A \in M_n.
\]
Therefore, we will only study the case when \( k > 1 \). It turns out that we also need to
impose some conditions on \( n \) and \( n' \) to avoid the pathetic situation described below.

Let \( k \in \{1, \ldots, n\} \), and \( (\alpha, \beta) \) be a pair of length \( k \) increasing subsequences of
\( \{1, \ldots, n\} \). Denote by \( A[\alpha, \beta] \) the submatrix of \( A \in M_n \) lying in rows and
columns indexed by \( \alpha \) and \( \beta \), respectively. Then the \( k \)th compound matrix of \( A \) is
the \( C(n, k) \times C(n, k) \) matrix \( C_k(A) \) whose entries equal \( \text{det} \, A[\alpha, \beta] \) arranged
in lexicographic order of \( \alpha \) and \( \beta \). The \( k \)th additive compound is defined by
\[ A_k(A) = \left. \frac{d}{dt} C_k(I + tA) \right|_{t=0}. \]

It is known (e.g., see [16]) that the mapping

\[ A \mapsto A_k(A) \]

from \( M_n \) to \( MC(n,k) \) is linear and satisfies

\[ W_k(A) = W_1(A_k(A)) \quad \text{for any } A \in M_n. \]

So, if \( n' \geq 2C(n,k) - 1 \) then there is a linear map \( \psi : MC(n,k) \to M_{n'} \) satisfying

\[ W_1(\psi(X)) = W_1(X) \quad \text{for all } X \in M_n \]

without nice structure. Thus, the linear map \( \phi : M_n \to M_{n'} \) defined by

\[ A \mapsto \psi(A_k(A)) \]

satisfies \( W_k(A) = W_1(\phi(A)) \) for all \( A \in M_n \) and does not have nice structure. For larger \( n' \) one can extend the above idea to construct \( \phi \) of the form

\[ A \mapsto \psi_1(A_k(A)) \oplus \cdots \oplus \psi_k(A_k(A)) \]

satisfying \( W_k(A) = W_k(\phi(A)) \) for all \( A \in M_n \) without nice structure.

By the above discussion, we see that it is reasonable to impose appropriate assumption on \( n, n', k, k' \) to obtain nice characterizations of linear map \( \phi : M_n \to M_{n'} \) satisfying \( W_k(A) = W_k(\phi(A)) \) for all \( A \in M_n \). This is done in Section 2. In fact, we show that the same result is valid for real linear map \( \phi : H_n \to H_{n'} \), where \( H_m \) denotes the real linear space of all \( m \times m \) complex Hermitian matrices. In Section 3, we extend the result to triangular matrices. Define the \( k \)-numerical radius of \( A \in M_n \) by

\[ w_k(A) = \max \{|z| : z \in W_k(A)\}. \]

In Section 4, we study those linear maps \( \tilde{\phi} \) satisfying

\[ w_k(\tilde{\phi}(A)) = w_k(A) \quad \text{for all } A \in M_n. \]

Some open problems are mentioned in Section 5.

Note that recently, researchers have also considered mappings preserving the classical numerical range and radius on more general operator algebras; see [1,2,4,5,7,8,14].

2. Results on Hermitian and complex matrices

The main theorem of this section is the following:

**Theorem 2.1.** Let \( (\mathcal{H}, \mathcal{H}') = (H_n, H_{n'}) \) or \( (M_n, M_{n'}) \). Suppose \( k \in \{2, \ldots, n-1\}, k' \in \{1, \ldots, n'\} \) and \( n' < C(n,k) \min\{k', n' - k'\} \). There exists a linear map \( \phi : \mathcal{H} \to \mathcal{H}' \) such that
\[ W_k'(\phi(A)) = W_k(A) \quad \text{for all } A \in \mathcal{M} \]  
(2.1)

if and only if there are unitary \( U \in \mathcal{M'} \) and nonnegative integers \( p, q \) with \( p + q = n'/n \) such that one of the following holds:

(W1) \( n'/n = k'/k \) and \( \phi \) has the form 
\[ A \mapsto U^n[A \oplus \cdots \oplus A \oplus A^t \oplus \cdots \oplus A^t]U. \]

(W2) \( n'/n = k'/(n-k) \) and \( \phi \) has the form 
\[ A \mapsto U^n[\psi(A) \oplus \cdots \oplus \psi(A) \oplus \psi(A)^t \oplus \cdots \oplus \psi(A)^t]U, \]

where \( \psi : \mathcal{M} \rightarrow \mathcal{M} \) is the mapping \( A \mapsto [(\text{tr} A)I_n - (n-k)A]/k. \)

Proof of the sufficiency part. Suppose \( n'/n = k'/k \). Then any mapping described in (W1) satisfies (2.1). If \( n'/n = k'/(n-k) \), then the mapping \( \phi : \mathcal{M} \rightarrow \mathcal{M'} \) described in (W2) satisfies
\[ W_k'(\phi(A)) = W_{n-k}(\psi(A)) = W_k(A) \quad \text{for all } A \in \mathcal{M}. \]  
□

In the following, we consider the converse. Suppose there exists a linear map \( \phi : \mathcal{M} \rightarrow \mathcal{M'} \) such that \( W_k'(\phi(A)) = W_k(A) \) for all \( A \in \mathcal{M} \). We will show that \( n'/n \) is an integer and one of conditions (W1) or (W2) holds by establishing a sequence of lemmas.

Let \( X \in \mathcal{H}_m \). Denote the eigenvalues of \( X \) by 
\[ \lambda_1(X) \geq \cdots \geq \lambda_m(X). \]

Suppose \( r \in \{1, \ldots, m-1\} \). Let
\[ S_r(X) = \sum_{j=1}^{r} \lambda_j(X) \quad \text{and} \quad s_r(X) = \sum_{j=1}^{r} \lambda_{m-j+1}(X). \]

We have the following result concerning the \( r \)-numerical range; see [11,12,20] and their references.

Lemma 2.2. Suppose \( r \in \{1, \ldots, m-1\} \).

(a) \( W_r(A) = W_r(U^*AU) \) for any unitary \( U \in \mathcal{M}_m \).
(b) \( W_r(\alpha A + \beta I_n) = \alpha W_r(A) + \beta \) for any \( \alpha, \beta \in \mathbb{C} \).
(c) If \( A \in \mathcal{H}_m \), then
\[ W_r(A) = [s_r(A)/r, S_r(A)/r]. \]
(d) A matrix \( B \in \mathcal{M}_m \) satisfies \( W_r(B) \subseteq \mathbb{R} \) if and only if \( B = B^* \).
(e) A matrix \( C \in \mathcal{M}_m \) satisfies \( W_r(C) = [\lambda] \) if and only if \( C = \lambda I_m \).
Lemma 2.3. The mapping $\phi$ satisfies $\phi(H_n) \subseteq H_{n'}$ and $\phi(I_n) = I_{n'}$.

Proof. If $A \in H_n$, then $W_k(\phi(A)) = W_k(A) \subseteq \mathbb{R}$. By Lemma 2.2 (d), $\phi(A) \in H_{n'}$. Furthermore, since $W_k(\phi(I_n)) = W_k(I_n) = \{1\}$, we have $\phi(I_n) = I_{n'}$ by Lemma 2.2 (e). □

By the above lemma, we can focus on proving the result for the Hermitian case. Once it is done, the result on complex matrices will follow from the fact that $\phi(A) = \phi(H) + i\phi(G)$ for any complex matrix $A = H + iG \in M_n$ with $H, G \in H_m$.

A key step in our proof is to show that $\phi$ or $\phi \circ \psi^{-1}$ will map idempotents in $H_n$ to idempotents in $H_{n'}$. Two idempotents $F, G \in H_m$ are said to be disjoint if $FG = GF = 0_m$.

Lemma 2.4. Suppose $A, B \in H_m$ and $r \in \{1, \ldots, m - 1\}$. The following conditions are equivalent.

(a) The sum of the first $r$ diagonal entries of $A + B$ equals $S_r(A) + S_r(B)$.
(b) The sum of the last $m - r$ diagonal entries of $A + B$ equals $s_{m-r}(A) + s_{m-r}(B)$.
(c) $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ such that $A_1$ has the $r$ largest eigenvalues of $A$ and $B_1$ has the $r$ largest eigenvalues of $B$.

Proof. Clearly, (a) and (b) are equivalent, and (c) implies (a). To prove (a) implies (c), let $d_A$ be the sum of the first $r$ diagonal entries of $A$, and $d_B$ be the sum of the first $r$ diagonal entries of $B$. Then $d_A \leq S_r(A)$ and $d_B \leq S_r(B)$. Now, the sum of the first $r$ diagonal entries of $A + B$ equals $d_A + d_B = S_r(A) + S_r(B)$. So, $d_A = S_r(A)$ and $d_B = S_r(B)$. By [13, Lemma 4.1], $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ with $A_1, B_1 \in M_r$ such that $A_1$ has the $r$ largest eigenvalues of $A$ and $B_1$ has the $r$ largest eigenvalues of $B$. □

By Lemma 2.4, one readily deduces the following.

Lemma 2.5. Suppose $A, B \in H_m$ and $1 < r < m$. The following conditions are equivalent.

(a) $S_r(A + B) = S_r(A) + S_r(B)$.
(b) $s_{m-r}(A + B) = s_{m-r}(A) + s_{m-r}(B)$.
(c) If $V \in M_m$ is unitary such that $V^*(A + B)V = \text{diag}(c_1, \ldots, c_m)$ with $c_1 \geq \cdots \geq c_m$, then $V^*AV = A_1 \oplus A_2$ and $V^*BV = B_1 \oplus B_2$ with $A_1, B_1 \in M_r$ such that $A_1$ (respectively, $B_1$) has the $r$ largest eigenvalues of $A$ (respectively, $B$).

Lemma 2.6. Suppose $k < n/2$ and $k' \leq n'/2$. If $E \in H_n$ is a rank one idempotent, then $\phi(E)$ is positive semidefinite.

Proof. Suppose $Q \in H_n$ is a rank $k$ idempotent such that $QE = EQ = 0$. Then $Q + E$ is a Hermitian idempotent with trace $k + 1$. Since $k < n/2$,
Then $\Sk k \phi(Q + E) = \Sk k(Q + E) = [0, 1]$, \\
$\Sk k \phi(Q) = \Sk k(Q) = [0, 1]$, and $\Sk k \phi(E) = \Sk k(E) = [0, 1/k]$. Then $S_k \phi(Q) = k'$, $s_k \phi(Q) = 0$, $S_k \phi(E) = k'/k$, and $s_k \phi(E) = 0$. So, 

$s_k \phi(Q) + \phi(E)) = s_k \phi(Q) + s_k \phi(E)).$

Let $V \in M_{k'}$ be unitary such that $V^* (\phi(Q + E))V = \text{diag}(c_1, \ldots, c_{n'})$ with $c_1 \geq \cdots \geq c_{n'}$. By Lemma 2.5, $V^* \phi(Q)V = Y_1 \oplus Y_2$ and $V^* \phi(E)V = Z_1 \oplus Z_2$ with $Y_2$ and $Z_2 \in M_{k'}$. Let $Z_2$ has the $k'$ smallest eigenvalues of $\phi(E)$. If $Z_2$ is the zero matrix, then $Z_1$ and hence, $\phi(E)$ is positive semidefinite as asserted. If $Z_2$ has a negative eigenvalue, then the largest eigenvalue of $Z_2$ is positive. Hence, $Z_1$ is positive definite. Now, suppose $V_1 \in M_{n' - k'}$ such that $V_1^* Y_1 V_1 = \text{diag}(b_1, \ldots, b_{n' - k'})$ with $b_1 \geq \cdots \geq b_{n' - k'}$. Then $\sum_{j=1}^{k'} b_j = s_k \phi(Q)) = k'$. Since $Z_1$ is positive definite and $k' \leq n'/2$, the sum of the first $k'$ diagonal entries of $V_1^* Z_1 V_1 = a > 0$. Let $X$ be the matrix consisting of the first $k'$ columns of the unitary matrix $V(V_1 \oplus I_{k'})$. Then 

$1 < (k' + a)/k' = \text{tr}(X^* \phi(Q + E)X)/k' \in W_{k'}(\phi(Q + E)),$

which contradicts the fact that $W_{k'}(\phi(Q + E)) = [0, 1]$. □

Lemma 2.7. Suppose $n' \leq n/2$ and $k' \leq n'/2$. Let $E_1, \ldots, E_n \in H_n$ be rank one idempotents such that $E_1 + \cdots + E_n = I_n$.

(a) Suppose $s \in \{1, \ldots, k' - 1\}$ such that 

$\lambda_{n'-s} \phi(E_1)) > \lambda_{n'-s+1} \phi(E_1))$.

Then there is an $n' \times s$ matrix $S$ whose columns are orthonormal eigenvectors of the eigenvalues $\lambda_{n'-s} \phi(E_1)), \ldots, \lambda_{n'} \phi(E_1))$ such that $S^* \phi(E_1)S = \gamma_{1}I_s$ with 

$\gamma_{1} = 1 - \sum_{j=2}^{n} \gamma_{j} = \lambda_{n'-s+1} \phi(E_1)) = \cdots = \lambda_{n'} \phi(E_1)),$

and 

$\gamma_{j} = \lambda_{n'-s+1} \phi(E_1)) = \cdots = \lambda_{n'-s+1} \phi(E_1)),$ \hspace{1cm} $j = 2, \ldots, n.$

(b) Suppose $r \in \{1, \ldots, k' - 1\}$ such that 

$\lambda_{r} \phi(E_1)) > \lambda_{r+1} \phi(E_1))$.

Then there is an $n' \times r$ matrix $R$ whose columns are orthonormal eigenvectors of the eigenvalues $\lambda_{1} \phi(E_1)), \ldots, \lambda_{r} \phi(E_1))$ such that $R^* \phi(E_1)R = \tilde{\gamma}_{1}I_r$ with

$\tilde{\gamma}_{1} = 1 - \sum_{j=2}^{n} \tilde{\gamma}_{j} = \lambda_{1} \phi(E_1)) = \cdots = \lambda_{r} \phi(E_1)),$
Because

\[ \tilde{\gamma}_j = \lambda_{k'-r+1}(\phi(E_j)) = \cdots = \lambda_{n'-k'+r}(\phi(E_j)), \quad j = 2, \ldots, n. \]

**Proof.** Assume that \( \phi(E_1) \) has eigenvalues \( a_1 \geq \cdots \geq a_{n'} \). Since

\[ W_k(\phi(E_1)) = W_k(E_1) = [0, 1/k], \]

\( a_{n'} + \cdots + a_{n'-k'+1} = 0 \). Let \( B \in \{E_2, \ldots, E_n\} \). Then

\[ W_k(\phi(B)) = W_k(B) = [0, 1/k] \]

and

\[ W_k(\phi(E_1 + B)) = W_k(E_1 + B) = [0, 2/k]. \]

Suppose \( \phi(B) \) has eigenvalues \( b_1 \geq \cdots \geq b_{n'} \).

(a) Note that \( s_k(\phi(E_1) + \phi(B)) = s_k(\phi(E_1)) + s_k(\phi(B)) \). By Lemma 2.5, there is a unitary \( V \in M_{n'} \) such that \( V^*\phi(E_1)V = A_1 \oplus A_2 \) and \( V^*\phi(B)V = B_1 \oplus B_2 \), where \( A_2 \) has eigenvalues \( a_{n'}, \ldots, a_{n'-k'+1} \) and \( B_2 \) has eigenvalues \( b_{n'}, \ldots, b_{n'-k'+1} \). Since \( a_{n'-s} > a_{n'-s+1} \), we may assume that the last \( s \) columns of \( V \) and \( V \) are the eigenvectors of \( \phi(E_1) \) corresponding to the eigenvalues \( a_{n'-s+1}, \ldots, a_{n'} \). So, the lower \( s \times s \) principal submatrices of \( B_2 \) and \( B_2 \) are the same, say, equal to \( X \in H_{s} \). Suppose \( X \) has eigenvalues \( d_1 \geq \cdots \geq d_{s} \). Because \( B_2 \) has eigenvalues \( b_{n'-k'+1} \geq \cdots \geq b_{n'} \), it follows from the interlacing inequalities (see [9]) that

\[ b_{n'-k'+j} \geq d_j, \quad j = 1, \ldots, s. \]  \hfill (2.2)

Because \( B_2 \) has eigenvalues \( b_1 \geq \cdots \geq b_{n'} \), by the interlacing inequalities again, we have

\[ d_j \geq b_{k'-s-j}, \quad j = 1, \ldots, s. \]  \hfill (2.3)

Since \( k' \leq n'/2, b_{k'-s-j} \geq b_{n'-k'+j} \) for \( 1 \leq j \leq s \). By (2.2) and (2.3), we see that

\[ b_{k'-s-j} = d_j = b_{n'-k'+j}, \quad j = 1, \ldots, s. \]

Thus,

\[ d_1 = \cdots = d_s = b_{k'-s+1} = \cdots = b_{n'-s+}. \]  \hfill (2.4)

Use the last \( s \) columns of \( V \) to form the matrix \( S \). Then \( S^*BS = d_1I_s \).

By the above arguments, \( S^*\phi(E_j)S = \gamma_jI_{s} \) for \( j = 2, \ldots, n \), where \( \gamma_j = \lambda_{k'-s+1}(\phi(E_j)) = \lambda_{n'-k'+s}(\phi(E_j)) \). By Lemma 2.3,

\[ \phi(E_1 + E_2 + \cdots + E_n) = \phi(I_n) = I_n'. \]
It follows that

\[ S^* \phi(E_j) S = I_z = \sum_{j=2}^n y_j I_z \]  

(2.5)

is a scalar matrix, where \( y_1 = 1 - \sum_{j=2}^n y_j \). Clearly, \( y_1 = a_{n'-s+1} = a_n' \).

(b) Note that

\[ W_k'(\phi(E_1 + B)) = W_k(E_1 + B) = [0, 2/k]. \]

Thus, \( S_k'\phi(E_1 + \phi(B)) = S_k\phi(E_1) + S_k\phi(B) \). Then there is a unitary \( W \in M_{n'} \) such that \( W^*\phi(E_j)W = Y_1 \oplus Y_2 \) and \( W^*\phi(B)W = Z_1 \oplus Z_2 \) with \( Y_1, Z_1 \in M_{k'} \), where \( Y_1 \) has eigenvalues \( a_1, \ldots, a_{k'} \) and \( Z_1 \) has eigenvalues \( b_1, \ldots, b_{n'} \). Now,

\[ W_k'(\phi(E_1 - B)) = W_k(E_1 - B) = [-1/k, 1/k]. \]

We see that \( S_k'(\phi(E_1 + \phi(-B)) = S_k\phi(E_1) + S_k\phi(-B) \). So there exists a unitary \( \tilde{W} \in M_{n'} \) such that \( W^*\phi(E_1)\tilde{W} = \tilde{Y}_1 \oplus \tilde{Y}_2 \) and \( W^*\phi(B)\tilde{W} = \tilde{Z}_1 \oplus \tilde{Z}_2 \), where \( \tilde{Y}_1 \) has eigenvalues \( a_1, \ldots, a_{k'} \) and \( \tilde{Z}_1 \) has eigenvalues \( b_{n'}, \ldots, b_{n'-k'+1} \). Since \( a_r > a_{r+1} \), we may assume that the first \( r \) columns of \( W \) and \( \tilde{W} \) are the eigenvectors of \( \phi(E_1) \) corresponding to the eigenvalues \( a_1, \ldots, a_r \). So, the leading \( r \times r \) submatrices of \( Z_1 \) and \( \tilde{Z}_1 \) are the same, say, equal to \( T \in H_r \). Suppose \( T \) has eigenvalues \( t_1 \geq \cdots \geq t_r \). Since \( Z_1 \) has eigenvalues \( b_1 \geq \cdots \geq b_{n'} \), by the interlacing inequalities

\[ t_j \geq b_{k'-r+j}, \quad j = 1, \ldots, r. \]

(2.6)

Since \( \tilde{Z}_1 \) has eigenvalues \( b_{n'-k'+1} \geq \cdots \geq b_{n'} \), by the interlacing inequalities

\[ b_{n'-k'+1} \geq t_j, \quad j = 1, \ldots, r. \]

(2.7)

Since \( k' \leq n'/2, b_{k'-r+j} \geq b_{n'-k'+j} \) for \( 1 \leq j \leq r \). By (2.6) and (2.7), we see that

\[ b_{k'-r+j} = t_j = b_{n'-k+j}, \quad j = 1, \ldots, r. \]

Since \( k' \leq n'/2, \)

\[ t_1 = \cdots = t_r = b_{k'-r+1} = \cdots = b_{n'-k'+r}. \]

Use the first \( r \) columns of \( W \) to form the matrix \( R \). Then \( R^T R = I_1 \). Consequently, \( R^*\phi(E_j)R = \gamma_j I_r \) for \( j = 2, \ldots, n \), as \( \gamma_j = \lambda_{k'-r+1}\phi(E_j) = \lambda_{n'-k'+r}\phi(E_j) \). Moreover, \( R^*\phi(E_1)R = I_r - \sum_{j=2}^n \gamma_j I_r \), where \( \gamma_1 = 1 - \sum_{j=2}^n \gamma_j \). Clearly, \( \gamma_1 = a_1 = a_r \).

Lemma 2.8. Suppose \( k = n/2 \) and \( k' \leq n'/2 \). If there is a rank one idempotent \( E \in H_n \) such that \( \phi(E) \) has negative eigenvalues, then \( k' = n'/2 \) and \( \phi \circ \psi^{-1}(F) \) is positive semidefinite for any rank one idempotent \( F \in H_n \).

Proof. Suppose there is a rank one idempotent \( E \) such that \( \phi(E) \) has negative eigenvalues. Assume that \( \phi(E) \) has eigenvalues \( a_1 \geq \cdots \geq a_{n'-s+1} \geq 0 > a_{n'-s+1} \geq \cdots \geq a_{n'} \). Since
By (2.9) we have
\[ W_k(\phi(E)) = W_k(E) = [0, 1/k], \]
\[ a_{n'} + \cdots + a_{n'-k'+1} = 0. \]
Thus, \( s < k' \).

Let \( E_1, \ldots, E_n \in H_k \) be rank one idempotents such that \( E_1 = E \) and \( \sum_{j=1}^{n} E_j = I_n \). By Lemma 2.7(a), there is an \( n' \times s \) matrix \( S \) whose columns are orthonormal eigenvectors of the negative eigenvalues of \( \phi(E_1) \) such that \( S^*\phi(E_j)S = \gamma_j I_s \) with
\[
\gamma_1 = 1 - \sum_{j=2}^{n} \gamma_j = a_{n'} = \cdots = a_{n'-s+1}
\]
(2.8)

and
\[
\gamma_j = \lambda_{k'-s+1}(\phi(E_j)) = \lambda_{n'-k'+s}(\phi(E_j)), \quad j = 2, \ldots, n.
\]
(2.9)

We must have
\[ a_1 = \cdots = a_{n'}. \]
Otherwise, there is \( r < k' \) such that \( a_r > a_{r+1} \). By Lemma 2.7(b), there is an \( n' \times r \) matrix \( R \) whose columns are orthonormal eigenvectors of the \( r \) largest eigenvalues of \( \phi(E_1) \) such that \( R^*\phi(E_j)R = \gamma_j I_r \) with
\[
\tilde{\gamma}_j = \lambda_{k'-r+1}(\phi(E_j)) = \lambda_{n'-k'+r}(\phi(E_j)), \quad j = 2, \ldots, n.
\]
(2.10)

By (2.9) and (2.10), we have \( \tilde{\gamma}_j = \lambda_{k'}(\phi(E_j)) = \gamma_j \) for \( j = 2, \ldots, n \). But then, we have \( R^*\phi(E_1)R = I_r - \sum_{j=2}^{n} \gamma_j I_r \), where \( a_1 = 1 - \sum_{j=2}^{n} \gamma_j = a_{n'} \), which is a contradiction. Since \( W_k(E_1) = W_k(\phi(E_1)) \), we see that \( a_1 = \cdots = a_{k'} = 1/k \).

Observe that \( \sum_{j=1}^{n} \gamma_j = 1 \) and \( \gamma_1 < 0 \). Thus, there exists \( j \geq 2 \) such that \( \gamma_j > 0 \).

We may assume that \( \gamma_2 > 0 \). Since \( s_k(\phi(E_2)) = 0 \) and \( \lambda_{n'-k'+s}(\phi(E_2)) = \gamma_2 > 0 \), we see that \( \lambda_{n'}(\phi(E_2)) < 0 \). Suppose \( \phi(E_2) \) has \( t \) negative eigenvalues. Applying the arguments on \( \phi(E_1) \) to \( \phi(E_2) \), we see that the last \( t \) eigenvalues of \( \phi(E_2) \) all equal to \( 1 - \sum_{j\neq 2}^{n} \lambda_{k'-s+1}(\phi(E_j)) \) and
\[ 1/k = \lambda_l(\phi(E_2)) \quad \text{for } l = 1, \ldots, k'. \]

By (2.9), we have
\[ 1/k = \gamma_2 = \lambda_{k'-s+1}(\phi(E_2)) = \cdots = \lambda_{n'-k'+s}(\phi(E_2)). \]

Interchanging the roles of \( E_1 \) and \( E_2 \), we see that
\[ 1/k = \lambda_1(\phi(E_1)) = \cdots = \lambda_{n'-k'+s}(\phi(E_1)). \]

Moreover, since \( \lambda_{k'}(\phi(E_1)) = \lambda_{k'}(\phi(E_2)) = 1/k \), we see that
\[ \lambda_{n'}(\phi(E_1)) = 1 - \sum_{j=3}^{n} \lambda_{k'}(\phi(E_j)) - 1/k = \lambda_{n'}(\phi(E_2)). \]

Suppose \( j \geq 3 \) is such that \( \phi(E_j) \) has negative eigenvalues. We can apply the above arguments on \( \phi(E_2) \) to \( \phi(E_j) \) to conclude that \( \lambda_{n'}(\phi(E_j)) = \gamma_1 \), and
\[ 1/k = \lambda_l(\phi(E_j)) \quad \text{for } l = 1, \ldots, n' - k' + s. \]
(2.11)
Suppose \( j \geq 3 \) and \( \phi(E_j) \) is positive semidefinite. Then \( \lambda_{n-k'+1}(\phi(E_j)) = \cdots = \lambda_n(\phi(E_j)) = 0 \). By (2.9), we have
\[
\lambda_{k'-s+1}(\phi(E_j)) = \lambda_{n-k'+s}(\phi(E_j)) = 0.
\]
(2.12)
Relabeling \( E_1, \ldots, E_n \) if necessary, we can assume that \( \phi(E_2), \ldots, \phi(E_m) \) have negative eigenvalues, and \( \phi(E_j) \) is positive semidefinite for \( j > m \). Then each one of \( \phi(E_1), \ldots, \phi(E_m) \) has smallest eigenvalue
\[
\gamma_1 = 1 - \sum_{j=1}^{n} \lambda_k(\phi(E_j)) = 1 - (m-1)/k < 0.
\]
If \( m < n \), then \( \phi(E_n) \) is positive semidefinite with fewer than \( k' \) positive eigenvalues. Since \( W_k(\phi(E_n)) = W_k(\phi(E_n)) = [0, 1/k] \), we see that \( \lambda_1(E_n) > 1/k \). By Lemma 5.7(b), we have
\[
1/k < \lambda_1(\phi(E_n)) = 1 - \sum_{j=1}^{n} \lambda_k(\phi(E_j)) = 1 - m/k < 1 - (m-1)/k < 0,
\]
which is a contradiction. So, we have \( m = n \). By (2.11), and the fact that \( W_k(\phi(E_j)) = W_k(\phi(E_j)) = [0, 1/k] \) for \( j = 1, \ldots, n \), we have
\[
n' = \text{tr}(I_n^c) = \sum_{j=1}^{n} \text{tr}(\phi(E_j))
= \sum_{j=1}^{n} S_k(\phi(E_j)) + \sum_{l=k'+1}^{n} \sum_{j=1}^{n} \lambda_l(\phi(E_j)) + \sum_{j=1}^{n} S_k(\phi(E_j))
= n(k'/k) + (n' - 2k')(1/k) = 2k' + 2(n' - 2k') = 2n' - 2k'.
\]
Hence, \( n' = 2k' \).

Suppose \( F \in H_n \) is a rank one idempotent. We claim that \( \lambda_1(F) = 1/k \). Since \( n = 2k \geq 4 \), there is a rank one idempotent \( G \in H_n \) such that \( EG = GE = 0 \) and \( FG = GF = 0 \). Moreover, there exist rank one idempotents \( G_1, \ldots, G_n \in H_n \) such that \( E + G + G_1 + \cdots + G_n = I_n \). Applying the previous argument with \( (E, E_2, \ldots, E_n) \) replaced by \( (G, G_1, \ldots, G_n) \), we see that \( G \) has negative eigenvalues. Now, there exist rank one idempotents \( F_1, \ldots, F_n \in H_n \) such that \( G + F + F_1 + \cdots + F_n = I_n \). Applying the previous arguments with \( (E, E_2, \ldots, E_n) \) replaced by \( (G, F, F_3, \ldots, F_n) \), we see that \( \phi(F) \) has largest eigenvalue \( 1/k \).

Now, observe that \( \psi^{-1}(F) = I_n/k - F \). Since \( \phi(F) \) has largest eigenvalue \( 1/k \), we conclude that \( \phi(\psi^{-1}(F)) = I_n/k - \phi(F) \), is positive semidefinite as asserted.

\[\square\]

**Lemma 2.9.** Suppose \( k \leq n/2, 2k' \leq n' < k' \cdot C(n, k) \), and \( \phi(E) \) is positive semidefinite for any rank one idempotent \( E \in H_n \). Then \( \phi(F) \) and \( \phi(G) \) are disjoint idempotents in \( H_{n'} \) for any disjoint rank one idempotents \( F, G \in H_n \).
Proof. Let $E_1, \ldots, E_n \in H_ν$ be rank one idempotents such that $F = E_1$, $G = E_2$, and $E_1 + \cdots + E_n = I_n$. Then $Y_j = φ(E_j) \in H_ν$ is positive semidefinite and

$$W_k(Y_j) = [0, 1/k] = [φ(Y_j)/k', S_k(Y_j)/k']$$

for all $j = 1, \ldots, n$. We claim that there is $j \in \{1, \ldots, n\}$ such that the largest eigenvalue of $Y_j$ has multiplicity $r < k'$. If it is not true, then for $j = 1, \ldots, n$,

$$\lambda_1(Y_j) = \cdots = \lambda_k(Y_j) = 1/k,$$

as $S_k(Y_j) = k'/k$. Now, for any $1 \leq j_1 < j_2 < \cdots < j_k \leq n$,

$$W_k\left(\sum_{t=1}^k Y_{j_t}\right) = W_k\left(\sum_{t=1}^k E_{j_t}\right) = [0, 1].$$

Thus, there exists an $n' \times k'$ matrix $U$ such that

$$k' = S_{k'}\left(\sum_{t=1}^k Y_{j_t}\right) = \text{tr}\left(U^*\left(\sum_{t=1}^k Y_{j_t}\right)U\right) = \sum_{t=1}^k \text{tr}(U^*Y_{j_t}U) \leq k'\sum_{t=1}^k S_k(Y_{j_t}) = k'.$$

It follows that $\text{tr}(U^*Y_jU) = k'/k$ and hence $U^*Y_jU = (1/k)I_k$ for $t = 1, \ldots, k$.

Since $Y_1 + \cdots + Y_n = I_n$, we see that $U^*Y_jU = 0_k$ for any $t \notin \{j_1, \ldots, j_k\}$.

Now, for any other choice of $1 \leq j_1 < j_2 < \cdots < j_k \leq n$, there is a corresponding $n' \times k'$ matrix $\tilde{U}$ such that $\tilde{U}^*\tilde{U} = I_{k'}$ such that $U^*Y_jU = (1/k)I_k$ for $t = 1, \ldots, k$, and $\tilde{U}^*Y_j\tilde{U} = 0_k$ for any $t \notin \{j_1, \ldots, j_k\}$. Suppose $\tilde{j}_p \notin \{j_1, \ldots, j_k\}$. Then $U^*Y_{j_p}U = 0_k$ and $U^*Y_{j_p}\tilde{U} = (1/k)I_{k'}$. Thus, the columns of $\tilde{U}$ belong to the kernel of $Y_{j_p}$ whereas the columns of $U$ belong to the kernel of $Y_{j_p} - (1/k)I_{k'}$.

So, $U^*\tilde{U} = 0_{k'}$.

Combining the above arguments, we see that there are $C(n, k)$ matrices $U$'s of size $n' \times k'$ such that $U^*U = I_{k'}$. Any two of such $U$ have mutually orthogonal columns. So, there are $k' \cdot C(n, k)$ orthonormal columns. Hence $k' \cdot C(n, k) \leq n'$, which contradicts our assumption.

By the above argument, we see that

$$\min\{r : \lambda_r(Y_t) > \lambda_{r+1}(Y_t) \text{ with } t \in \{1, \ldots, n\}\} < k'.$$

Relabeling $Y_1, \ldots, Y_n$ if necessary, we may assume that

$$\lambda_1(Y_1) = \cdots = \lambda_r(Y_1) > \lambda_{r+1}(Y_1)$$

and for $t = 2, \ldots, n$,

$$\lambda_1(Y_t) = \cdots = \lambda_r(Y_t).$$

Note that the last $k'$ eigenvalues of $Y_j$ are all zeros. We claim that the first $k'$ eigenvalues of $Y_t$ can contain at most two distinct values. Otherwise, there are $1 \leq s <
s′ < k′ such that λs(Yt) > λs+1(Yt) and λs′(Yt) > λs′+1(Yt). But by Lemma 2.7(b), λ1(Yt) = · · · = λk(Yt), which is impossible.

Note that the last k′ eigenvalues of Yt are all zeros. Applying Lemma 2.7(b) to Y1, we have

λk−r+1(Yt) = λk′−k+r(Yt) = 0

for t = 2, . . . , n. Then there is r1 < k′ − r + 1 ≤ k′ such that

λ1(Yt) = λr1(Yt) > λr1+1(Yt) = λr′(Yt) = 0,

i.e., Yt is unitarily similar to γtIt1 ⊕ 0n′−r1 for t = 2, . . . , n. Interchanging the role of Y1 and Yt, we conclude that Y1 is unitarily similar to γ1It1 ⊕ 0n′−r1. Since Wk(Yt) = Wk(Et) = [0, 1/k], r1γt = k′/k.

Furthermore, we can see from Lemma 2.7(b) that for s ̸= t, all eigenvectors of Ys corresponding to the eigenvalue γs are eigenvectors of Yt corresponding to the eigenvalue 0. Hence, YsYt = 0 for any s ̸= t. Since Y1 + · · · + Yn = In′, γ1 = 1 and r1 + · · · + rn = n′. Hence, r1 = k′/k = r for all t = 1, . . . , n and k′/k = n′/n. This shows that every Yt is unitarily similar to It ⊕ 0n′−r. Hence, A ϕ (A) maps disjoint idempotents to disjoint idempotents. □

Proof of the necessity part of Theorem 1. Suppose k < n/2 and k′ ≤ n′/2. By Lemmas 2.6 and 2.9, ϕ will map idempotents to idempotents. So, (see [10, Corollary 4.3] and also [3, Theorem 2.1]), ϕ has the asserted form.

Suppose k = n/2 and k′ ≤ n′/2. Apply Lemma 2.8; then apply Lemmas 2.6 and 2.9 to ϕ ◦ ψ−1 to get the conclusion.

Suppose k > n/2 and k′ ≤ n′/2. Then ϕ ◦ ψ−1 satisfies Wn−k(A) = Wk(ϕ ◦ ψ−1(A)) for all A ∈ Mn. So, ϕ ◦ ψ−1 has the desired form.

Suppose k > n/2 and k′ > n′/2. Replace ϕ by ψ ◦ ϕ ◦ ψ−1 with ψ: Mn′ → Mn′ defined by ψ(X) = [(trX)It − k′X]/(n′ − k′) for all X ∈ Mn′. Then Wn−k(A) = Wn′−k′(ϕ(A)) for all A ∈ Mn. So, ψ ◦ ϕ ◦ ψ−1 has the asserted form. It follows that ϕ has the same form as well. □

3. Results on triangular matrices

Let Tn be the set of n × n upper triangular matrices. In this section, we study those linear maps ϕ : Tn → Tn satisfying

Wk′(ϕ(A)) = Wk(A) for all A ∈ Tn. (3.1)

Clearly, if a map ϕ has the form (W1) or (W2) in Theorem 2.1 for some unitary U such that ϕ(Tn) ⊆ Tn′, then condition (3.1) holds. The following theorem shows that the converse of the above statement is also valid, and gives a condition on U to ensure that ϕ(Tn) ⊆ Tn′.
Theorem 3.1. Suppose \( k \in \{2, \ldots, n-1\} \), \( k' \in \{1, \ldots, n'\} \) and \( n' < C(n, k) \min\{k', n' - k'\} \). There exists a linear map \( \phi : T_n \to T_{n'} \) such that
\[
W_k(\phi(A)) = W_k(A) \quad \text{for all} \quad A \in T_n
\]
if and only if there are unitary \( U = (u_{ij}) \in M_{n'} \) and nonnegative integers \( p, q \) with \( p + q = n'/n \) such that
\[
\sum_{j=0}^{p-1} u_{(jn+a),d} \bar{u}_{(jn+b),c} + \sum_{j=p}^{p+q-1} u_{(jn+b),d} \bar{u}_{(jn+a),c} = 0 \quad (3.2)
\]
for all \( 1 \leq a \leq b \leq n \) and \( 1 \leq c < d \leq n' \), and one of the following holds:

(T1) \( n'/n = k'/k \) and \( \phi \) has the form
\[
A \mapsto U^* [A \oplus \cdots \oplus A \oplus A^t \oplus \cdots \oplus A^t] U.
\]

(T2) \( n'/n = k'/(n-k) \) and \( \phi \) has the form
\[
A \mapsto U^* [\psi(A) \oplus \cdots \oplus \psi(A) \oplus \psi(A)^t \oplus \cdots \oplus \psi(A)^t] U,
\]
where \( \psi : T_n \to T_n \) is the mapping \( A \mapsto [(\text{tr} A)I_n - (n-k)A]/k \).

Let us further analyze condition (3.2) in the following. For any \( 1 \leq a \leq b \leq n \) and \( 1 \leq c \leq n' \), define
\[
\bar{u}_{bc}^a = \begin{pmatrix} u_{bc}^a \\ u_{(p-1)n+b)c} \\ \vdots \\ u_{(p+q-1)n+a)c} \end{pmatrix}
\]
Then (3.2) reduces to
\[
(v_c^d, v_c^a) + (u_{d}^b, u_{c}^a) = 0
\]
for all \( 1 \leq a \leq b \leq n \) and \( 1 \leq c < d \leq n' \),
where \((\cdot, \cdot)\) denotes the usual inner product, i.e., \( (x, y) = y^* x \). Suppose \( U = (u_{ij}) \) satisfies (3.2). Clearly, the set \( \{u_{1a}^{1a}, \ldots, u_{na}^{na}\} \) forms an orthogonal set on \( \mathbb{C}^{n'/n} \). Then at most \( n'/n \) vectors of the set can be nonzero. Hence, at most \( n(n'/n) = n' \) vectors of the set can be nonzero. As \( U \) is an \( n' \times n' \) unitary matrix, exactly one vector in \( \{u_{1a}^{1a}, \ldots, u_{na}^{na}\} \) can be nonzero. Otherwise, \( U \) has a zero column. Furthermore, if \( a \neq b \),
then at most one of $v_c^b$ and $w_c^a$ can be nonzero, as only one of $u_c^{bb}$ and $u_c^{aa}$ can be nonzero. Thus, we deduce from (3.2) that

$$(v_c^a, v_c^b) = (u_c^a, u_c^b) \quad \text{for all } 1 \leq a < b \leq n \text{ and } 1 \leq c < d \leq n'.$$

In conclusion, we have the following.

**Proposition 3.2.** A unitary matrix $U = (u_{ij}) \in M_{n'}$ satisfies (3.2) if and only if

(i) for each $1 \leq c \leq n'$, there is an $a \in \{1, \ldots, n\}$ such that $$(v_c^a, v_c^a) + (w_c^a, w_c^a) = 1$$ and $v_c^b$ and $w_c^b$ are zero vectors for all $b \neq a$; and

(ii) for any $1 \leq a < b \leq n$ and $1 \leq c < d \leq n'$,

$$(v_a^c, v_b^c) = (v_a^d, v_b^d) = (w_a^c, w_b^c) = 0.$$ 

**Example.** If $n' = 6$, $n = 2$ and $q = 1$, then

$$U = \begin{pmatrix}
1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} & 0 \\
0 & 0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\
1/\sqrt{2} & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\
0 & 0 & 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

satisfies (3.2). But if $U_1$ is the matrix obtained from $U$ by interchanging its $(4, 4)$th and $(4, 6)$th entries, then $U_1$ does not satisfy (3.2) as $(v_2^4, v_2^5) = 1$.

**Proof of Theorem 3.1.** Note that the left side of (3.2) equals the $(d, c)$th entry of

$$U^*[E_{1}\oplus \cdots \oplus E_{1}\oplus E_{b}\oplus \cdots \oplus E_{b}]U,$$

where $E_{ij}$ is the matrix in $M_n$ with 1 at the $(i, j)$ entry and 0 otherwise.

Hence, (3.2) holds if and only if

$$U^*[A\oplus \cdots \oplus A\oplus A\oplus \cdots \oplus A]U \in T_{n'} \quad \text{for all } A \in T_n.$$ 

Therefore, if $\phi$ has the form (T1), $\phi(T_n) \subseteq T_{n'}$. If $\phi$ has the form (T2), since $\psi(T_n) = T_n$, we have the same conclusion too. Therefore the sufficiency part holds.

For the converse, take any diagonal matrix $R$ with real diagonal entries. Then

$$W_k(\phi(R)) = W_k(R) \subseteq \mathbb{R}.$$ 

By Lemma 2.2(d), $\phi(R)^* = \phi(R)$. As $\phi(R) \in T_{n'}$, $\phi(R)$ must be a diagonal matrix with real diagonal entries.
Now for any diagonal matrix \( D \in D_n \), write \( D_1 = (D + D^*)/2 \) and \( D_2 = (D - D^*)/(2i) \). Then \( D_1 \) and \( D_2 \) are diagonal matrices with real diagonal entries. It follows that \( \phi(D_1) = \phi(D_1) \) and \( \phi(D_2) = \phi(D_2) \). As \( D = D_1 + iD_2 \),

\[
\phi(D^*) = (\phi(D_1) - i\phi(D_2))^* = \phi(D_1)^* + i\phi(D_2)^* = \phi(D_1) + i\phi(D_2) = \phi(D).
\]

Every \( A \in M_n \) can be expressed as \( T_1 + T_2^* \) for some upper triangular matrices \( T_1 \) and \( T_2 \). Define \( \Phi : M_n \rightarrow M_n' \) by

\[
\Phi(A) = \Phi(T_1) + \Phi(T_2)^*.
\]

Clearly, \( \Phi \) is linear. Suppose \( A \) can be written as \( U_1 + U_2^* \) for some \( U_1, U_2 \in T_n \) distinct from \( T_1 \) and \( T_2 \). Let \( D = T_1 - U_1 = U_2^* - T_2^* \). Then \( D \) is a diagonal matrix. Observe that

\[
0 = \phi(D) - \phi(D^*) = \phi(T_1 - U_1) - \phi(U_2 - T_2)^* = \phi(T_1) + \phi(T_2)^* - \phi(U_1) - \phi(U_2)^* = \phi(T_1 + T_2^*) - \phi(U_1 + U_2^*).
\]

Hence, \( \Phi \) is well defined. On the other hand, we see that for any \( A \in M_m \) and \( 1 \leq r \leq m \),

\[
W_r(A + A^*) = \{ \text{tr}(X^*AX)/r + \text{tr}(X^*A^*X)/r : X \text{ is } m \times r, X^*X = I_r \} = \{ \text{tr}(X^*AX)/r + \text{tr}(X^*A^*X)/r : X \text{ is } m \times r, X^*X = I_r \} = \{ z + \bar{z} : z \in W_r(A) \}.
\]

Since every matrix \( H \in H_n \) can be expressed as \( H = T + T^* \) with \( T \in T_n \),

\[
W_k(\Phi(H)) = W_k(\phi(T) + \phi(T)^*) = \{ z + \bar{z} : z \in W_k(\phi(T)) \} = \{ z + \bar{z} : z \in W_k(T) \} = W_k(T + T^*) = W_k(H).
\]

Hence, \( \Phi : M_n \rightarrow M_n' \) is a linear map such that

\[
W_k(\Phi(H)) = W_k(H) \quad \text{for all } H \in H_n.
\]

By Theorem 2.1, there exist a unitary \( U \in M_n' \) and nonnegative integers \( p, q \) with \( p + q = n'/n \) such that \( \Phi \) satisfies (W1) or (W2) in Theorem 2.1. Since \( \phi(A) = \Phi(A) \) for all \( A \in T_n \), \( \phi \) has the form (T1) or (T2). Finally, we check that \( U \) satisfies (3.2) as \( \phi(E_{ab}) \in T_{n'} \) for all \( a \leq b \).

**Remark 3.3.** Denote by \( T(n_1, \ldots, n_r) \) the algebra of upper block triangular matrices \( A = (A_{ij}) \) such that \( A_{ii} \in M_{n_i} \) for \( i = 1, \ldots, r \). One can extend Theorem 3.1 to linear map \( \phi : T(n_1, \ldots, n_r) \rightarrow T(m_1, \ldots, m_s) \) for \( n_1 + \cdots + n_r = n, m_1 + \cdots + m_s = n', \) and \( n' < C(n, k) \text{min} \{k', n' - k'\} \). The result and proofs are basically the
same provided that $U$ satisfies (3.2) for all $1 \leq a, b \leq n$, $1 \leq c, d \leq n'$ such that $E_{ab} \in T(n_1, \ldots, n_r)$ and $E_{cd} \notin T(m_1, \ldots, m_s)$. Since the corresponding statements are rather tedious, we omit the details.

Note also that if a linear map $\phi : T(n_1, \ldots, n_r) \rightarrow T(m_1, \ldots, m_s)$ satisfies $W_k(\phi(A)) = W_k(A)$ for all $A \in T(n_1, \ldots, n_k)$, then one can replace $T(m_1, \ldots, m_s)$ by other block triangular matrix algebras such as $T(m_1 + m_2, m_3, \ldots, m_s)$, etc.

4. $k$-Numerical radius

**Theorem 4.1.** Let $\mathcal{H} = (H_n, H_{n'})$, $(M_n, M_{n'})$ or $(T_n, T_{n'})$, $k \in \{2, \ldots, n - 1\}$, $k' \in \{1, \ldots, n'\}$ and $n' < C(n, k) \min\{k', n' - k'\}$. Then a linear operator $\tilde{\phi} : \mathcal{H} \rightarrow \mathcal{H'}$ satisfies $W_{k'}(\tilde{\phi}(A)) = W_{k'}(A)$ for all $A \in \mathcal{H}$, and $\tilde{\phi}(X) = I_{n'}$ for some $X \in \mathcal{H}$ if and only if there is a complex unit $\mu$ such that $\phi = \mu \tilde{\phi}$ satisfies $W_{k'}(\phi(A)) = W_{k'}(A)$ for all $A \in \mathcal{H}$, equivalently, $\phi$ has the form in Theorem 2.1 or Theorem 3.1.

**Lemma 4.2.** For any $T = (t_{ij}) \in T_n$, if

$$\frac{1}{k} \left| \sum_{i=1}^{k} t_{in_i, n_i} \right| = w_k(T) \text{ for all } 1 \leq n_1 < \cdots < n_k \leq n,$$

then $T$ is a diagonal matrix.

**Proof.** Suppose $t_{ij} \neq 0$ for some $i < j$. Denote by $X[i, j] \in M_2$ the submatrix of $X \in M_n$ lying in the rows and columns indexed by $i$ and $j$. Then $W_1(T[i, j])$ is an elliptical disk with the length of minor axis equal to $|t_{ij}|$, and foci $t_{ii}$ and $t_{jj}$; see [11]. Thus, there is a unitary $U \in M_2$ such that the $(1, 1)$ entry of $U^*T[i, j]U$ equals $t_{ii} + z$ and

$$|z + t_{ii} + \sum_{i=2}^{k} t_{in_i, n_i}| > |t_{ii} + \sum_{i=2}^{k} t_{n_i, n_i}| = k w_k(T),$$

where $1 \leq n_2 < \cdots < n_k \leq n$ are chosen from $\{1, \ldots, n\} \setminus \{i, j\}$. Let $V \in M_n$ be obtained from $I_n$ by replacing $I_n[i, j]$ with $U$, and $V^*TV = (t_{rs})$. Then

$$k w_k(T) = k w_k(V^*TV) \geq \left| t_{ii} + \sum_{i=2}^{k} t_{n_i, n_i} \right| = |z + t_{ii} + \sum_{i=2}^{k} t_{n_i, n_i}| > k w_k(T),$$

which is a contradiction. \qed
Lemma 4.3. Let \( \mathcal{A} = H_n, M_n \) or \( T_n \). Suppose \( k \in \{1, \ldots, n-1\} \). Given any matrix \( A \in \mathcal{A} \), \( A = \mu I_n \) with \( |\mu| = 1 \) if and only if for any \( B \in \mathcal{A} \), there is \( \theta \) (depending on \( B \)) with \( |\theta| = 1 \) such that

\[
wk(A + \theta B) = wk(A) + wk(B) = 1 + wk(B),
\]

i.e., there is an \( n \times k \) matrix \( U \) (depending on \( B \)) with \( U^*U = I_k \) such that

\[
wk(A) = |\text{tr}(U^*AU)|/k = 1 \quad \text{and} \quad wk(B) = |\text{tr}(U^*BU)|/k. \tag{4.3}
\]

Proof. Suppose \( A = \mu I_n \) for some \( |\mu| = 1 \). For any \( B \in \mathcal{A} \), if \( wk(B) = |\text{tr}(U^*BU)|/k \) for some \( n \times k \) matrix \( U \) with \( U^*U = I_k \), then

\[
wk(B) = |\text{tr}(U^*AU)|/k = |\text{tr}(U^*U_\mu I_n U)|/k = 1 = wk(A).
\]

For the converse, suppose for any \( B \in \mathcal{A} \), there is an \( n \times k \) matrix \( U \) with \( U^*U = I_k \) such that (4.3) holds.

Let \( K = I_k \oplus 0_{n-k} \). For any \( n \times k \) matrix \( X \) with \( X^*X = I_k \), we write \( X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \) with \( X_1 \in M_k \). Clearly, \( \text{tr}(X_1^*X_1 + X_2^*X_2) = \text{tr}(X^*X) = k \). Then

\[
(k - \text{tr}(X_1^*X_1))/k = \text{tr}(X_1^*X_1)/k = \text{tr}(X^*KX)/k \in W_k(K) = [0, 1].
\]

It follows that \( \text{tr}(X^*KX)/k = 1 = wk(K) \) if and only if \( X_1^*X_2 = 0 \). Since \( X_1^*X_2 \) is positive semidefinite, \( \text{tr}(X_1^*X_1) = 0 \) if and only if \( X_2^*X_2 = 0 \). Thus, \( X_1 \) must be unitary.

Suppose \( \mathcal{A} = H_n \) or \( M_n \). Take any \( n \times k \) matrix \( V \) with \( V^*V = I_k \), we extend \( V \) to an \( n \times n \) unitary matrix \( W = (V \quad V') \) with some suitable \( n \times (n-k) \) matrix \( V' \). Choose \( B = WKW^* \). Then there is an \( n \times k \) matrix \( U \) with \( U^*U = I_k \) such that

\[
1 = wk(A) = |\text{tr}(U^*AU)|/k \quad \text{and} \quad wk(K) = wk(WKW^*) = |\text{tr}(U^*WKW^*U)|/k.
\]

By the above argument, \( W^*U = X = \begin{pmatrix} X_1 \\ 0 \end{pmatrix} \) for some unitary matrix \( X_1 \in M_k \). Thus, \( U = VX_1 \) and

\[
|\text{tr}(V^*AV)/k = |\text{tr}(X_1^*V^*AVX_1)/k| = |\text{tr}(U^*AU)|/k = wk(A) = 1.
\]

It follows that all elements of \( W_k(A) \) lie on the unit circle. Since \( W_k(A) \) is convex, \( W_k(A) \) must be a singleton set. By Lemma 2.2(e), \( A = \mu I_n \) for some \( |\mu| = 1 \).

It remains to show the case for \( \mathcal{A} = T_n \). Let \( T = A \). For any \( 1 \leq n_1 < \cdots < n_k \leq n \), let \( P = (p_{ij}) \) be the \( n \times n \) permutation matrix with \( p_{n_i,j} = 1 \) for \( i = 1, \ldots, k \) and \( B = PKP \). Then there is an \( n \times k \) matrix \( U \) with \( U^*U = I_k \) such that

\[
1 = wk(T) = |\text{tr}(U^*TU)|/k \quad \text{and} \quad wk(K) = wk(PKP^*) = |\text{tr}(U^*PKP^*U)|/k.
\]
By the above argument, \( P^*U = X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \) for some unitary matrix \( X_1 \in M_k \). Thus,

\[
k_w(T) = |\text{tr}(U^*TU)| = |\text{tr}(X^*P^*PX)|
\]

\[
= |\text{tr}(X_1^*T_1X_1)| = |\text{tr} T_1| = \left| \sum_{i=1}^k t_{n_i} \right|
\]

where \( T_1 \) is the \( k \times k \) principal submatrix of \( P^*P \). As \( n_1, \ldots, n_k \) are arbitrary, \( T \) satisfies (4.2). By Lemma 4.2, we conclude that \( T \) is a diagonal matrix.

Finally we show that the diagonal entries of \( T \) are the same. Suppose \( t_{ii} / t_{jj} \neq t_{jj} / t_{ii} \) for some \( i \neq j \). For simplicity, we assume that \( t_{11} / t_{22} \). Take \( B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \oplus I_{k-1} \oplus 0_{n-k-1} \). Then \( w_k(B) = 1 \) and \( |\text{tr}(X^*BX)|/k = w_k(B) \) if and only if the \( n \times k \) matrix \( X \) has the form \( \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \) with \( X_1 = \left( \alpha / \sqrt{2} \right) \) for some \( |\alpha| = 1 \) and unitary \( X_2 \in M_{k-1} \). In this case,

\[
\left| \frac{1}{2}(t_{11} + t_{22}) + \sum_{i=3}^{k+1} t_{ii} \right| = |\text{tr}(X^*X)| = k.
\]

Let \( \alpha = t_{11} + \sum_{i=3}^{k+1} t_{ii} \) and \( \beta = t_{22} + \sum_{i=3}^{k+1} t_{ii} \). Since \( T \) satisfies (4.2), we see that

\[
|(\alpha + \beta)/2| = k = |\alpha| = |\beta|,
\]

and hence \( t_{11} = t_{22} \), which is the desired contradiction. \( \square \)

The following lemma is a modification of [15, Lemma 2], we give the proof here for the sake of completeness.

**Lemma 4.4.** Let \( (\mathcal{M}, \mathcal{M}') = (H_n, H_n'), (M_n, M_n') \text{ or } (T_n, T_n') \), \( k \in \{1, \ldots, n-1\} \) and \( k' \in \{1, \ldots, n'\} \). If \( \hat{\phi} : \mathcal{M} \to \mathcal{M}' \) is a linear map satisfying (4.1) and \( \hat{\phi}(I_n) = I_{n'} \), then

\[
W_k(\hat{\phi}(A)) = W_k(A) \text{ for all } A \in \mathcal{M}.
\]

**Proof.** Suppose \( W_k(A) \not\subseteq W_k(\hat{\phi}(A)) \). Let \( z \in W_k(A) \setminus W_k(\hat{\phi}(A)) \). Since \( W_k(\hat{\phi}(A)) \) is compact, there exists some \( \lambda \in \mathbb{C} \) such that

\[
|z + \lambda| > |z' + \lambda|
\]

for all \( z' \in W_k(\hat{\phi}(A)) \). Here,

\[
w_k(A + \lambda I_n) > w_k(\hat{\phi}(A) + \lambda I_n') = w_k(\hat{\phi}(A + \lambda I_n)) = w_k(A + \lambda I_n)
\]

which is impossible. Therefore, \( W_k(A) \subseteq W_k(\hat{\phi}(A)) \). Similarly, we have \( W_{k'}(\hat{\phi}(A)) \subseteq W_k(A) \). The result follows. \( \square \)
Proof of Theorem 4.1. The sufficiency part is clear. For the necessity part, suppose there is $X \in \mathcal{M}$ such that $\tilde{\phi}(X) = I_{n'}$. For any $B \in \mathcal{M}$, there exists some $\theta \in \mathbb{C}$ with $|\theta| = 1$ such that

$$w_k(X + \theta B) = w_k(I_{n'} + \theta \tilde{\phi}(B)) = w_k(I_{n'}) + w_k(\tilde{\phi}(B)) = w_k(X) + w_k(B).$$

By Lemma 4.3, $X = \mu I_n$ for some $\mu \in \mathbb{C}$ with $|\mu| = 1$. We see that the map $A \mapsto \mu \tilde{\phi}(A)$ maps $I_n$ to $I_{n'}$ and satisfies (4.3). Then the result follows by Lemma 4.4. □

5. Open problems

There are many open problems deserved further study. We mention a few of them in the following.

1. If $n' = C(n, k) \min\{k', n' - k'\}$, there are exceptional maps for the range preservers having the form

$$A \mapsto U^* \Delta_k(A) U \text{ or } A \mapsto U^* \Delta_k(A)^t U$$

with $k' = 1$. Are there other exceptional maps?

2. If $n' \leq 2C(n, k) \min\{k', n' - k'\} - 2$, there are exceptional maps for the range preservers having the form

$$A \mapsto U^*[\Delta_k(A) \oplus f(\Delta_k(A))]U \text{ or } A \mapsto U^*[\Delta_k(A)^t \oplus f(\Delta_k(A))]U$$

for some unital positive linear map $f: M_{C(n, k)} \to M_{n' - C(n, k)}$, here $k' = 1$. Are there other exceptional maps?

3. In Theorem 4.1, an assumption that $\tilde{\phi}(X) = I_{n'}$ for some $X \in \mathcal{M}$ is needed. For $k' = 1$, since $w_1(A) = w_1(A \oplus 0)$, the condition is clearly necessary. Can this assumption be removed when $k' > 1$?

4. How about extending the results to infinite dimensional operators, nest algebras, etc.?

5. What about other types of generalized numerical ranges and radii?

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