ISOMETRIES FOR KY FAN NORMS
BETWEEN MATRIX SPACES

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Abstract. We characterize linear maps between different rectangular matrix
spaces preserving Ky Fan norms.

1. Introduction and statements of results

Let $M_{m,n}$ ($M_n$) be the linear space of $m \times n$ ($n \times n$) complex matrices. The
singular values of $A \in M_{m,n}$ are the nonnegative square roots of the eigenvalues of
$A^*A$, and they are denoted by $s_1(A) \geq \cdots \geq s_n(A)$. For $1 \leq k \leq \min\{m,n\}$, the
Ky Fan $k$-norm on $M_{m,n}$ is defined and denoted by
$$
\|A\|_k = s_1(A) + \cdots + s_k(A).
$$

The Ky Fan 1-norm reduces to the operator norm when $m = n$ the Ky Fan $n$-norm
is also known as the trace norm.

Evidently, Ky Fan $k$-norms are unitarily invariant norms, i.e.,
$$
\|UAV\|_k = \|A\|_k
$$
for any $A \in M_{m,n}$, and unitary $U \in M_m$ and $V \in M_n$. Actually, they form an
important class of unitarily invariant norms; see [1, Chapters 2 and 3]. For instance,
given $A, B \in M_{m,n}$,
$$
\|A\|_k \leq \|B\|_k \quad \text{for all } k = 1, \ldots, \min\{m,n\}
$$
if and only if
$$
\|A\| \leq \|B\| \quad \text{for all unitarily invariant norms } \| \cdot \|.
$$

There has been considerable interest in studying isometries for Ky Fan norms on
matrix spaces. For example, by a result of Kadison [5], one easily deduces that
isometries for the operator norm on $M_n$ have to have the form
(1) \hspace{1cm} A \mapsto UAV \quad \text{or} \quad A \mapsto UAV

for some unitary matrices $U, V \in M_n$. In [4], the authors showed that the same
conclusion holds for Ky Fan $k$-norm isometries for any $k = 1, \ldots, \min\{m,n\}$, where
the second form in (1) can occur only when $m = n$. In [8], the authors considered the

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problem on block triangular matrix algebras in $M_n$, and showed that the isometries essentially have the same structure. In [3], the authors studied isometries $\phi : (M_n, \| \cdot \|_1) \rightarrow (M_p, \| \cdot \|_1)$ for $n \neq p$, and obtained a complete characterization when $p \leq 2n - 2$; moreover, examples were given to show that $\phi$ may have complicated structure for $p > 2n - 2$. In view of these, one may think that isometries $\phi : (M_n, \| \cdot \|_k) \rightarrow (M_p, \| \cdot \|_k)$ also have complicated structure for $k > 1$. It turns out that it is not the case as shown in the corollary of our main theorem, which characterizes isometries $\phi : (M_{m,n}, \| \cdot \|_{k'}) \rightarrow (M_{p,q}, \| \cdot \|_{k'})$ provided $k' > 1$. We need some notation and definitions to describe our main result.

For two matrices $A$ and $B$ with $A = (a_{ij})$ denote by $A \otimes B$ the block matrix $(a_{ij}B)$. An $r \times s$ matrix $X$ is called a partial isometry if $X^* X = I_s$, i.e., $X$ has orthonormal columns.

**Theorem 1.1.** Let $1 < k' \leq \min\{m, n\}$ and $1 \leq k \leq \min\{p, q\}$. Suppose $\phi : M_{m,n} \rightarrow M_{p,q}$ satisfies

$$\|\phi(A)\|_k = \|A\|_{k'} \quad \text{for all} \quad A \in M_{m,n}. \tag{2}$$

Then there exist nonnegative integers $c_1$ and $c_2$ with $c_1 + c_2 > 0$, and partial isometries $U$ and $V$ of sizes $p \times (c_1 m + c_2 n)$ and $q \times (c_1 n + c_2 m)$, respectively, such that one of the following holds:

(a) $k' < \min\{m,n\}$, $k = k'(c_1 + c_2)$, and $\phi$ has the form

$$A \mapsto \frac{1}{c_1 + c_2} U[(I_{c_1} \otimes A) + (I_{c_2} \otimes A)^t] V^*.$$  

(b) $k' = \min\{m,n\}$, $k'(c_1 + c_2) \leq k$, and there are diagonal matrices $D_1 \in M_{c_1}$ and $D_2 \in M_{c_2}$ with positive diagonal entries satisfying $\text{tr} D_1 + \text{tr} D_2 = 1$, such that $\phi$ has the form

$$A \mapsto U[(D_1 \otimes A) + (D_2 \otimes A^t)] V^*.$$  

If $k' = k$, then either $(c_1, c_2) = (1, 0)$ or $(c_1, c_2) = (0, 1)$. By adding columns to $U$ and $V$ to form unitary matrices, we have the following corollary.

**Corollary 1.2.** Let $1 < k \leq \min\{m,n\}$. Suppose $\phi : M_{m,n} \rightarrow M_{p,q}$ satisfies

$$\|\phi(A)\|_k = \|A\|_k \quad \text{for all} \quad A \in M_{m,n}. \tag{3}$$

Then there are unitary matrices $U \in M_p$ and $V \in M_q$ such that $\phi$ has the form

$$A \mapsto U[A \oplus 0_{p-m,n-a} + V \quad \text{or} \quad A \mapsto U[A^t \oplus 0_{p-n,q-m} + V].$$

2. Auxiliary results and proofs

Replacing $\phi$ by the mapping(s) $A \mapsto \phi(A^t)$ and/or $A \mapsto [\phi(A)]^t$, we may assume that $m \leq n$ and $p \leq q$. Two nonzero matrices $A, B \in M_{m,n}$ are said to be orthogonal if $AB^* = 0$ and $A^*B = 0$. Equivalently, there are unitary matrices $U$ and $V$ such that $UAV = \sum_{j=1}^r a_{ij}E_{jj}$ and $UBV = \sum_{j=r+1}^{r+s} b_{ij}E_{jj}$ with $a_{ij} \geq \cdots \geq a_r > 0$ and $b_{ij} \geq \cdots \geq b_s > 0$ for some $r, s$ with $r + s \leq \min\{m,n\}$. The nonzero matrices $A_1, \ldots, A_d \in M_{m,n}$ are said to be pairwise orthogonal $m \times n$ matrices if $A_i A_j^* = 0$ and $A_i^* A_j = 0$ for any distinct pair $(i, j)$. In such a case, there are unitary $U \in U_m$ and $V \in U_n$, $0 = r_0 < r_1 < \cdots < r_d \leq \min\{m,n\}$ and positive numbers $a_1, \ldots, a_{rd}$ such that $UA_i V = \sum_{r_{i-1} < j \leq r_i} a_j E_{jj}$.  

We begin with the following lemma from [8] Lemma 5.

**Lemma 2.1.** Let \( A, B \in M_{m,n} \) be nonzero. Then \( \|aA + bB\|_k = |a|\|A\|_k + |b|\|B\|_k \) for every \( a, b \in \mathbb{C} \) and only if \( A \) and \( B \) are orthogonal and \( \text{rank} \, A + \text{rank} \, B \leq k \).

By Lemma 2.1 and a simple inductive argument, we have the following.

**Lemma 2.2.** Let \( \phi : M_{m,n} \to M_{p,q} \) be a map satisfying (2). Suppose the rank one matrices \( A_1, \ldots, A_d \in M_{m,n} \), \( d \leq \min\{m,n\} \), are pairwise orthogonal. Then \( \phi(A_1), \ldots, \phi(A_d) \in M_{p,q} \) are nonzero and pairwise orthogonal. Furthermore, for any \( 1 \leq s < \cdots < s_k \leq d \), \( \sum_{j=1}^{k'} \text{rank} \, \phi(A_{s_j}) \leq k \).

**Proof of Theorem 1.1.** For the sufficiency part of Theorem 1.1 one readily sees that singular values of \( \phi(A) \) have \( c = (c_1 + c_2) \) copies of \( s_1(A)/c, \ldots, s_m(A)/c \) if \( \phi \) has the form (a). On the other hand, if \( k' = m \) and \( \phi \) has the form (b), then \( k \geq ck' \) and so the Ky Fan \( k \)-norm of \( \phi(A) \) is just the sum of its singular values.

Let \( D_1 \oplus D_2 = \text{diag}(d_1, \ldots, d_c) \). Then, \[
\|\phi(A)\|_k = d_1\|A\|_{k'} + \cdots + d_c\|A\|_{k'} = \text{tr}(D_1 \oplus D_2)\|A\|_{k'} = \|A\|_{k'}.
\]

To prove the necessity part, let \((p', q') = (p - c_1m - c_2n, q - c_1n - c_2m)\). It suffices to prove that there are unitary matrices \( U \in M_p \) and \( V \in M_q \) such that \( \phi \) has the form

\[
(a) \quad A \mapsto \frac{1}{c_1 + c_2} U[(I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A^t) \oplus 0_{p', q'}]V^* \quad \text{if} \quad k' < m,
\]

\[
(b) \quad A \mapsto U[(D_1 \otimes A) \oplus (D_2 \otimes A^t) \oplus 0_{p', q'}]V^* \quad \text{if} \quad k' = m.
\]

We divide the proof into three cases:

(I) \( k' < m = n \),  \quad (II) \( k' = m = n \),  \quad and  \quad (III) \( m < n \).

First consider case (I) \( k' < m = n \). For any \( A \in M_{m,n} \) with singular values \( 1, 0, \ldots, 0 \), there are unitary \( X \) and \( Y \) such that \( A = XE_{11}Y \). Let \( A_j = XE_{jj}Y \) for \( j = 1, \ldots, m \). Then \( A_1, \ldots, A_m \) are pairwise orthogonal. By Lemma 2.2 \( \phi(A_1), \ldots, \phi(A_m) \) are pairwise orthogonal. Thus, there exist unitary \( U \) and \( V \), \( 0 = r_0 < r_1 < \cdots < r_d \leq m \) and positive numbers \( a_1, \ldots, a_r \) such that

\[
B_i = U\phi(A_i)V = \sum_{r_{i-1} < j \leq r_i} a_j E_{jj} \quad \text{for any} \quad i = 1, \ldots, m.
\]

By Lemma 2.2 again, the sum of any \( k' \) matrices chosen from \( B_1, \ldots, B_m \) has rank at most \( k \). Let \( 1 \leq t_1 < \cdots < t_{k'} \leq m \). Then

\[
(3) \quad s_\ell \left( \sum_{j=1}^{k'} B_{t_j} \right) = 0, \quad \text{for all} \quad \ell > k.
\]

Moreover, if \( t \in \{1, \ldots, m\} \setminus \{t_1, \ldots, t_{k'}\} \), we claim that

\[
(4) \quad s_1(B_t) \leq s_k \left( \sum_{j=1}^{k'} B_{t_j} \right).
\]

If (4) does not hold, then \( s_1(B_t) > s_k \left( \sum_{j=1}^{k'} B_{t_j} \right) \), which gives the following contradiction:

\[
k' = \left\| A_t + \sum_{j=1}^{k'} A_{t_j} \right\|_{k'} = \left\| B_t + \sum_{j=1}^{k'} B_{t_j} \right\|_k = \left\| \sum_{j=1}^{k'} B_{t_j} \right\|_k = k'.
\]
Let $c = k / k'$. It follows from (2), (3) and (4) that for each $1 \leq j \leq m$, we have $s_i(B_j) = 1/c$ for $1 \leq i \leq c$ and $s_i(B_j) = 0$ for $c < i \leq p$. Thus, we see that

(i) every rank one matrix is mapped to a rank $c$ matrix, and

(ii) every unitary matrix is mapped to a matrix with the singular values $1/c, \ldots, 1/c, 0, \ldots, 0$.

Since (i) holds, by Theorem 2.5 in [27], $\phi$ has the form

$$A \mapsto R_{k}(I_{c_1} \otimes A) + (I_{c_2} \otimes A^t) \oplus 0_{p', q'}|S^*$$

for some invertible $R \in M_p$ and $I \in M_q$. Let $R_1$ (respectively, $S_1$) be obtained from $R$ (respectively, $I$) by removing its last $p'$ (respectively, $q'$) columns. Then

$$R_{k}(I_{c_1} \otimes A) + (I_{c_2} \otimes A^t) \oplus 0_{p', q'}|S^* = R_{k}(I_{c_1} \otimes A) + (I_{c_2} \otimes A^t)|S_1^*.$$

By polar decomposition, there are unitary matrices $U \in M_p, V \in M_q$ and positive definite matrices $P \in M_{c_1 m + c_2 n}$ and $Q \in M_{c_1 n + c_2 m}$ such that

$$R_1 = U \begin{pmatrix} P \\ 0_{p', c_1 m + c_2 n} \end{pmatrix} \quad \text{and} \quad S_2 = V \begin{pmatrix} Q \\ 0_{q', c_1 n + c_2 m} \end{pmatrix}.$$

Thus,

$$\phi(A) = U \{ P[(I_{c_1} \otimes A) + (I_{c_2} \otimes A^t)]Q^* + 0_{p', q'} \} V^*.$$

Define $\psi : M_{m} \to M_{cm}$ such that $\psi(X) = cP[(I_{c_1} \otimes A) + (I_{c_2} \otimes A^t)]Q^*$. By (ii), we see that $\psi$ maps unitary matrices to unitary matrices. By the result in [27], we see that $\psi(A) = W_1[(I_{c_1} \otimes A) + (I_{c_2} \otimes A^t)]W_2$ for some unitary $W_1, W_2 \in M_{cm}$. Thus, condition (a) holds.

Next, we turn to case (II) : $k' = m = n$. From the first part of the proof in case (I), we can see that for any unitary $X, Y \in M_m$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$, $\sum_{i=1}^m \lambda_i \phi(X E_{ii} Y)$ has rank at most $k$. Hence, $\phi(A)$ has rank at most $k$ for all $A \in M_m$. We may assume that $p = q$ by appending $q - p$ zero rows to $\phi(A)$ for each $A \in M_m$. So, we assume that $\phi : M_m \to M_p$ and suppose $\phi(I_m) = D$ is a nonnegative diagonal matrix with diagonal entries arranged in descending order. For any Hermitian $X \in M_m$ with trace zero and spectrum in $[-1, 1]$ and $t \in [-1, 1]$,

$$\|\phi(I_m + tX)\|_k = \|I_m + tX\|_{k'} = k' = \|I_m\|_{k'} = \|\phi(I_m)\|_k = \operatorname{tr}D.$$

Let $Y = \phi(X)$. Then $\operatorname{tr}Y = 0$ because

$$\operatorname{tr}D + t\operatorname{tr}Y \leq \|\phi(I_m + tX)\|_p = \|\phi(I_m + tX)\|_k = \operatorname{tr}D$$

for $t = \pm 1$. Moreover,

$$k' = \operatorname{tr}(D \pm Y) \leq \|\phi(I_m + tX)\|_p = \|\phi(I_m + tX)\|_k = k'.$$

By [3] Corollary 3.2, we conclude that $D \pm Y$ is positive semidefinite. As a result, if $\phi(I_{m}) = D = \text{diag}(d_1, \ldots, d_r, 0, \ldots, 0)$ with $d_1 \geq \cdots \geq d_r > 0$, then $\phi(X)$ has the form $Y \oplus 0_{p-r}$. We may now consider $\psi : M_m \to M_p$ such that $\phi(A) = \psi(A) \oplus 0_{p-r}$. It follows from the above argument that $\psi$ maps Hermitian matrices to Hermitian matrices and $\|\psi(A)\|_p = \|\phi(A)\|_k = \|A\|_{k'}$. We claim that

(i) $\psi$ maps positive semidefinite matrices to positive semidefinite matrices, and

(ii) $\psi$ maps invertible Hermitian matrices to invertible Hermitian matrices.
To see (i), suppose that $A \in M_m$ is positive semidefinite. Let $D_1 = \psi(I_m) = \text{diag}(d_1, \ldots, d_r)$. Choose $t > 0$ such that $D_1 + t\psi(A)$ is positive semidefinite. Then we have
\[
\text{tr}(D_1 + t\psi(A)) = \|D_1 + t\psi(A)\|_F = \|I_m + tA\|_{\text{tr}} = \text{tr}(I_m) + t\text{tr}(A)
\]
\[
= \|I_m\|_{\text{tr}} + t\|A\|_{\text{tr}} = \|\psi(I_m)\|_r + t\|\psi(A)\|_r = \text{tr}D_1 + t\|\psi(A)\|_r.
\]
Thus, $\text{tr}\psi(A) = \|\psi(A)\|_r$, and it follows from [6, Corollary 3.2] again that $\psi(A)$ is positive semidefinite.

To prove (ii), let
\[
A = U^* \left( \sum_{j=1}^{m} \lambda_j E_{jj} \right) U
\]
for some unitary $U$ and $\lambda_j \in \mathbb{R} \setminus \{0\}$ for $j = 1, \ldots, m$. Since $\phi(U^*E_{11}U), \ldots, \phi(U^*E_{mm}U)$ are pairwise orthogonal and $\phi(I_m) = D$, $\phi(U^*E_{jj}U) = V^*F_jV \oplus 0_{r-j}$, for $j = 1, \ldots, m$, such that $F_j = \sum_{r_{j-1} < s \leq r_j} a_{ss}E_{ss}$ for $0 = r_0 < \cdots < r_m = r$ and positive numbers $a_1, \ldots, a_r$. Therefore,
\[
\psi(A) = V^* \left( \sum_{j=1}^{m} \lambda_j \left( \sum_{r_{j-1} < s \leq r_j} a_{ss}E_{ss} \right) \right) V
\]
is also invertible. Thus, condition (ii) holds.

Now, $\psi(I_m)$ is positive definite and $\psi$ maps invertible Hermitian matrices to invertible Hermitian matrices. By (the proof of) [7, Proposition 3.4], we see that
\[
\psi(X) = T^*[(I_{c_1} \otimes X) \oplus (I_{c_2} \otimes X^t)]T
\]
for some invertible $T \in M_r$. In particular, we see that

(iii) $\psi$ maps rank $s$ matrices to rank $cs$ matrices for $s = 1, \ldots, m$.

Next, we show that $\psi$ has the form $X \mapsto U^*[(D_1 \otimes X) \oplus (D_2 \otimes X^t)]U$ for some unitary matrix $U$ and diagonal matrices $D_1$ and $D_2$ with positive diagonal entries such that $\text{tr}D_1 + \text{tr}D_2 = 1$. Equivalently, we show that $\psi$ has the form
\[
A = (a_{uv}) \mapsto V^*BV, \text{ where } B = (B_{uv})_{1 \leq u,v \leq m} \text{ with } B_{uv} = a_{uv}D_1 \oplus a_{uv}D_2
\]
for some unitary $V$. First, by a suitable permutation, we can rewrite $\psi$ in (5) as
\[
A = (a_{uv}) \mapsto S^*BS \text{ with } B = (B_{uv})_{1 \leq u,v \leq m} \text{ with } B_{uv} = a_{uv}I_{c_1} \oplus a_{uv}I_{c_2}
\]
for some nonsingular $S \in M_r$. By Lemma 2.2, we see that $\phi(E_{11}), \ldots, \phi(E_{mm})$ are pairwise orthogonal. Then for any distinct pair $i$ and $j$,
\[
[S^*(E_{ii} \otimes I_c)]^* [S^*(E_{jj} \otimes I_c)]S = \psi(E_{ii})^*\psi(E_{jj}) = 0.
\]
Thus, $(E_{ii} \otimes I_c)S^*E_{jj} \otimes I_c = 0$ whenever $i \neq j$. It follows that $SS^* = S_1 \oplus \cdots \oplus S_n$ where $S_i \in M_{c_i}$.

Let $i > 1$, $X = E_{i1} + E_{ii}$ and $Y = E_{i1} - E_{ii}$. From (6), $\psi(X) = S^*(B_{ri})S$ and
\[
\psi(Y) = S^*(C_{ri})S \text{ so that }
\]
\[
\tilde{B} = \begin{pmatrix} B_{11} & B_{1i} \\ B_{i1} & B_{ii} \end{pmatrix} = \begin{pmatrix} I_c & I_{c_1} \oplus 0_{c_2} \\ 0_{c_1} \oplus I_{c_2} & 0_c \end{pmatrix},
\]
\[
\tilde{C} = \begin{pmatrix} C_{11} & C_{1i} \\ C_{i1} & C_{ii} \end{pmatrix} = \begin{pmatrix} 0_c & 0_{c_1} \oplus I_{c_2} \\ I_{c_1} \oplus 0_{c_2} & -I_c \end{pmatrix}.
\]
and all other $B_{uv}$ and $C_{uv}$ are 0. Let $J_1 = I_{c_1} \oplus 0_{c_2}$ and $J_2 = 0_{c_1} \oplus I_{c_2}$. Since $X$ and $Y$ are orthogonal, so are $\psi(X)$ and $\psi(Y)$. Hence $B^*(SS^*)C = 0$ and $B(SS^*)C^* = 0$. Thus,

$$
\begin{pmatrix}
J_2 S_1 J_1 & S_1 J_2 - J_2 S_1 \\
0 & S_1 J_1 - J_1 S_1
\end{pmatrix} = \tilde{B}^*(S_1 \oplus S_i) \tilde{C}^* = 0
= \tilde{B}(S_1 \oplus S_i) \tilde{C}^* = \begin{pmatrix}
J_1 S_1 J_2 & S_1 J_1 - J_1 S_1 \\
0 & J_2 S_1 J_1
\end{pmatrix}.
$$

Since $J_2 S_1 J_1 = J_1 S_1 J_2 = J_2 S_1 J_1 = J_1 S_1 J_2 = 0$, each of the matrices $S_1$ and $S_i$ is a direct sum of a matrix in $M_{c_1}$ and a matrix in $M_{c_2}$. Furthermore, we can conclude that $S_1 = S_i = P_1 \oplus P_2$, where $P_1 \in M_{c_1}$ and $P_2 \in M_{c_2}$, from $S_1 J_1 - J_1 S_1 = 0 = S_1 J_2 - J_2 S_1$. Since $i$ is arbitrary, $SS^* = I_m \otimes (P_1 \oplus P_2)$ where $P_1$ and $P_2$ are both positive definite. Thus there exist unitary $U_1 \in M_{c_1}$ and $U_2 \in M_{c_2}$ such that $U_1 P_1 U_1^* = D_1$ and $U_2 P_2 U_2^* = D_2$, where $D_1$ and $D_2$ are diagonal matrices with positive diagonal entries.

Let $U = I_m \otimes (U_1 \oplus U_2)$ and $\tilde{S} = US$. Then $\tilde{S}^* S = I_m \otimes (D_1 \oplus D_2)$. Since the row vectors of $\tilde{S}$ form an orthogonal basis, we may write $\tilde{S} = DV$, where $D = I_m \otimes (D_1 \oplus D_2)^{1/2}$ and $V$ is unitary.

On the other hand, we have $U^* B U = B$ for the block matrix $B$ in (1), since

$$
a_{uv} I_{c_1} \oplus a_{uv} I_{c_2} = (U_1 \oplus U_2)^* (a_{uv} I_{c_1} \oplus a_{uv} I_{c_2}) (U_1 \oplus U_2).
$$

Then $S^* B S = S^* U^* BUS = \tilde{S}^* B \tilde{S} = V^* D^* B D V$. In fact, the $(i, j)$-th block of $D^* BD$ is equal to

$$
(D_1 + D_2)^{1/2} (a_{uv} I_{c_1} \oplus a_{uv} I_{c_2}) (D_1 + D_2)^{1/2} = a_{uv} D_1 + a_{uv} D_2.
$$

Thus, $\phi$ has the asserted form. Since $\|I_m \otimes (D_1 \oplus D_2)\|_{k'} = \|\psi(I_m)\|_r = \|I_m\|_{k'} = m$, it follows that $\text{tr}(D_1 \oplus D_2) = \text{tr} D_1 + \text{tr} D_2 = 1$.

Finally, we consider case (III): $m < n$. We prove the desired conclusion by induction on $n - m$ starting from $n - m = 0$, which follows from cases (I) and (II). Suppose that $n - m = r > 0$ and that the result holds for the cases when $n - m < r$. Applying the assumption on the restriction of $\phi$ on $M^0_{m,n}$, the subspace of $M_{m,n}$ that consists of matrices with zero $n$-th column, we conclude that for any $A \in M^0_{m,n}$,

$$
\phi(A) = U[(D_1 \otimes \tilde{A}) \oplus (D_2 \otimes \tilde{A}^t) \oplus 0_{p',q'}] V
$$

where $\tilde{A}$ denotes the $m \times (n - 1)$ matrix obtained from $A$ by deleting the $n$-th column, $(p', q') = (p - c_1 m - c_2 (n - 1), q - c_1 (n - 1) - c_2 m)$, $U \in M_p$ and $V \in M_q$ are unitary, and the following holds:

(a) if $k' < m$ and $c = c_1 + c_2 = k'/k'$, then $D_1 = I_{c_1}$ and $D_2 = I_{c_2}$;
(b) if $k' = m$ and $c = c_1 + c_2 \leq k'/k'$, then $D_1 \in M_{c_1}$ and $D_2 \in M_{c_2}$ are diagonal matrices with positive diagonal entries such that $\text{tr} D_1 + \text{tr} D_2 = 1$.

Now replacing $\phi$ by $X \mapsto U^* \phi(X) V^*$, we may assume that $U = I_p$ and $V = I_q$.

For any $x \in M_{m,1}$, let $A$ be the $m \times n$ matrix with $x$ as the $n$-th column and zero in the other columns, and $X = [X_{uv}]_{1 \leq u \leq c_1, 1 \leq v \leq c_2} = \phi(A)$, where $X_{uv} \in M_{m,n-1}$ for $1 \leq u \leq c_1$, $X_{uv} \in M_{m-1,m}$ for $c_1 < u \leq c$ and $X_{v \leq c_1, c_2 + 1} \in M_{p',q'}$.

Take any nonzero $y \in M_{m,1}$ such that $x^* y = 0$. (Note that $1 < k \leq m$ and hence $y$ exists.) For any $l < n$, let $B$ be the $m \times n$ matrix with $y$ as the $l$-th column and zero in all other columns. Then $Y = \phi(B) = (D_1 \otimes \tilde{B}) \oplus (D_2 \otimes \tilde{B}^t) \oplus 0_{p',q'}$. 


Since \( A \) and \( B \) are orthogonal, \( X^*Y = 0_q \) and \( XY^* = 0_p \). It follows from the structure of \( Y \) that

\[
\begin{align*}
X_{uv}^* \tilde{B} &= 0 \quad \text{when} \quad 1 \leq u \leq c_1 \text{ and } 1 \leq v \leq c + 1, \\
X_{uv}^* B_t &= 0 \quad \text{when} \quad c_1 < u \leq c \text{ and } 1 \leq v \leq c + 1, \\
X_{uv}^* B^t &= 0 \quad \text{when} \quad 1 \leq u \leq c + 1 \text{ and } 1 \leq v \leq c_1, \\
X_{uv}(B^t)^* &= 0 \quad \text{when} \quad 1 \leq u \leq c + 1 \text{ and } c_1 < v \leq c.
\end{align*}
\]

Since the \( l \)-th column of the \( m \times (n - 1) \) matrix \( \tilde{B} \) is the nonzero vector \( y \), if \( X_{uv}^* B^* = 0 \), then the \( l \)-th row of \( X_{uv} \) must be the zero vector. Furthermore, since \( l \) can be any integer in \( \{1, \ldots, n - 1\} \), we conclude that \( X_{uv} = 0 \). Similarly, \( X_{uv} \) must be the zero matrix if \( X_{uv}^* B^t = 0 \).

On the other hand, if \( X_{uv}^* B = 0 \), then all the columns of \( X_{uv} \) must be orthogonal to \( y \). Since \( y \) can be any vector orthogonal to \( x \), all columns of \( X_{uv} \) must be multiples of \( x \). Hence, \( X_{uv} = xw^t \) for some vector \( w \) of suitable size. Similarly, since \( X_{uv}(B^t)^* = 0 \), we have \( X_{uv} = xz^t \) for some \( z \).

By the arguments in the last two paragraphs, if \( 1 \leq u \leq c_1 \) and \( c_1 < v \leq c \), then \( xw^t = X_{uv} = xz^t \) for some \( w \) and \( z \) of suitable sizes. Thus, \( w = \lambda x \) for some constant \( \lambda \) in \( \mathbb{C} \), that is, \( X_{uv} = \lambda xx^t \).

Combining the above analysis, we know that

\[
\phi[0_{m, n - 1} | x] = \begin{pmatrix} 0_{c_1, m, c_1, n} & E(x) & F(x) \\ 0_{c_2, n, c_1, 1} & 0_{c_2, n, c_2, m} & 0_{c_2, m, q, c_1} \\ 0_{p', c_1, n} & G(x) & H(x) \end{pmatrix}
\]

where

\[
\begin{align*}
E(x) &= (\lambda_{uv} xx^t)^{1 \leq u \leq c_1, 1 \leq v \leq c_2}, \\
F(x) &= \begin{pmatrix} xw^t_1 \\ \vdots \\ xw^t_{c_1} \end{pmatrix}, \\
G(x) &= (z_1 w^t \cdots z_{c_2} x^t),
\end{align*}
\]

\( H(x), \lambda_{uv}, w_u \) and \( z_v \) all depend on \( x \). By linearity of \( \phi \), \( \lambda_{uv} \), \( w_u \) and \( z_v \) must be the same for all \( x \), and \( \lambda_{uv} \) must be zero. i.e., \( E(x) = 0_{c_1, m, c_2, m} \).

Now we consider the orthogonal pair \( A = E_{11} + E_{1n} \) and \( B = -E_{21} + E_{2n} \). Let \( e_i \) be the \( i \)-th column of \( I_m \). Then

\[
\phi(A) = \begin{pmatrix} D_1 \otimes \tilde{E}_{11} & 0_{c_1, m, c_2, m} & F(e_1) \\ 0_{c_2, n, c_1} & D_2 \otimes \tilde{E}_{11} & 0_{c_2, m, q, c_1} \\ 0_{p', c_1, n} & G(e_1) & H(e_1) \end{pmatrix}
\]

and

\[
\phi(B) = \begin{pmatrix} D_1 \otimes -\tilde{E}_{21} & 0_{c_1, m, c_2, m} & F(e_2) \\ 0_{c_2, n, c_1} & D_2 \otimes -\tilde{E}_{21} & 0_{c_2, m, q, c_1} \\ 0_{p', c_1, n} & G(e_2) & H(e_2) \end{pmatrix}.
\]
Set \( W = \begin{pmatrix} w_1^t \\ \vdots \\ w_n^t \end{pmatrix} \). Since \( \phi(A)\phi(B)^* = 0 \), the \((1,1)\)-th block equals

\[
0_{c_1 m} = (D_1 \otimes \bar{E}_{11})(D_1 \otimes -\bar{E}_{21})^* + F(e_1)F(e_2)^* \\
= -(D_2^2 \otimes E_{12}) + (WW^* \otimes E_{12}) \\
= (WW^* - D_1^2) \otimes E_{12}.
\]

Thus, \( WW^* = D_1^2 \). Let \( D_1 = \text{diag} (d_1, \ldots, d_{c_1}) \). Hence, \( \{w_1/d_1, \ldots, w_{c_1}/d_{c_1}\} \) is a set of orthonormal vectors. Let \( U \in M_{q'} \) be a unitary matrix with \( w_{c_1}^t/d_{c_1} \) as the first \( c_1 \) rows. Then \( F'(x) = F(x)U^* = [D_1 \otimes x | 0_{c_1 m, q'-c_1}] \).

Similarly, by considering \( \phi(A)^*\phi(B) = 0 \), we write

\[
G'(x) = V^*G(x) = \begin{pmatrix} D_2 \otimes x^t \\ 0_{p'-c_2, c_2 m} \end{pmatrix}
\]

for some unitary \( V \). Now, we write

\[
\phi[0_{m,n-1} | x] = (I_{cn} \oplus V) \begin{pmatrix} 0_{c_1 m, c_1 n} & 0_{c_1 m, c_2 m} & F'(x) \\ 0_{c_2 n, c_1 n} & 0_{c_2 n, c_2 m} & 0_{c_2 n, q'} \\ 0_{p', c_1 n} & G'(x) & H'(x) \end{pmatrix} (I_{cn} \oplus U).
\]

On the other hand, by applying the assumption on the restriction of \( \phi \) on the subspace of \( M_{m,n} \) that consists of matrices with zero in the \((n-1)\)-th column, we conclude that

\[
\text{rank } \phi[0_{m,n-1} | x] = \text{rank } \phi[x | 0_{m,n-2} | x] = \text{rank } \phi[x | 0_{m,n-1}] = c.
\]

(Note that here we used the fact that \( n > m \geq 2 \) to ensure nontrivial consideration.) Therefore, \( H'(x) = 0 \) for all \( x \). Finally, there exist permutation matrices \( P \) and \( Q \) such that for \( \bar{A} = [0_{m,n-1} | x] \),

\[
\phi(A) = (I_{cn} \oplus V)P[D_1 \otimes A \oplus (D_2 \otimes A^t) \oplus 0_{p'-c_2, q'-c_1}]Q(I_{cn} \oplus U).
\]

The result follows. \( \square \)

References


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