STABLE RANK OF SOME CROSSED PRODUCT $C^*$-ALGEBRAS

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ABSTRACT. Let $C(X) \times_T Z$ be the crossed product associated to a dynamical system $(X, T)$. We give a necessary and sufficient condition for $C(X) \times_T Z$ to have a dense set of invertible elements. When $X$ is zero-dimensional, we obtain more equivalent conditions which involve the isomorphism between the $K$-groups of $C(X) \times_T Z$ and $C^*$-algebras defined by some $T$-invariant closed subsets of $X$. As an application, we show that these conditions are not satisfied by most subshifts and all nontrivial irreducible Markov shifts. When $(X, T)$ is indecomposable, an equivalent condition is that the intersection of all $T$-invariant nonempty closed subsets of $X$ is nonempty.

1. INTRODUCTION

Given a unital $C^*$-algebra $A$, let $\text{Lg}_n(A)$ be the set of $n$-tuples in $A^n$ which generates $A$ as a left ideal. The topological stable rank of $A$ is defined (Rieffel [15]) as the smallest integer $n$ such that $\text{Lg}_n(A)$ is dense in $A^n$. If no such $n$ exists, the topological stable rank of $A$ is defined to be $\infty$. For simplicity, we will just call this the stable rank of $A$, $sr(A)$. If $A$ does not have a unit, then $sr(A)$ is defined to be $sr(\tilde{A})$, where $\tilde{A}$ is the $C^*$-algebra obtained from $A$ by adjoining a unit [8]. One of the reasons for studying stable rank is to obtain cancellation theorems for the classification of projective modules over $A$ (e.g. Rieffel [16], Sheu [19]). Thus, given a $C^*$-algebra $A$, one would like to determine $sr(A)$. In particular, $sr(A) = 1$ if and only if the invertible elements are dense in $A$. This case has attracted a lot of attention [2, 6, 7, 12, 15, 17, 18]. One of the nice properties of these $C^*$-algebras is that they all have cancellation for projections [1, 6.4.1]. In this note, we will study the stable rank of the crossed product associated to some dynamical systems.

A (dynamical) system $(X, T)$ consists of a compact space $X$ and a homeomorphism $T$ on $X$. Given a system $(X, T)$, we have an action of the integers $Z$ on $C(X)$, the $C^*$-algebra of complex continuous functions on $X$. This gives a crossed product $C(X) \times_T Z$ [8], which is a $C^*$-algebra generated by

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C(X) and a unitary U satisfying $UfU^* = f \circ T^{-1}$ for $f \in C(X)$. If $Y$ is a nonempty $T$-invariant closed subset of $X$, then we have a system $(Y, T)$. By restricting the functions in $C(X)$ to $Y$, we have a $C^*$-homomorphism $\pi$ from $C(X) \times_T Z$ onto $C(Y) \times_T Z$. Let $I_y$ be the kernel of $\pi$. Our first result is that $\text{sr}(C(X) \times_T Z) = 1$ if and only if $\text{sr}(I_y) = \text{sr}(C(Y) \times_T Z) = 1$ and the homomorphism $\pi_y: K_1(C(X) \times_T Z) \to K_1(C(Y) \times_T Z)$ is onto. Then we restrict our attention to systems $(X, T)$ where $X$ is zero-dimensional. A compact metrizable space is said to be zero-dimensional if the topology on $X$ has a basis of sets which are both closed and open (clopen). For such systems, $\text{sr}(C(X) \times_T Z)$ is either 1 or 2 (Rieffel [15, Theorem 7.1]). By computing the $K_1$-groups explicitly, we derive some necessary conditions on $(X, T)$ for $\text{sr}(C(X) \times_T Z) = 1$. An application of this result shows that for most subshifts and all nontrivial irreducible Markov shifts [4], $\text{sr}(C(X) \times_T Z) = 2$. A system $(X, T)$ is said to be minimal if $X$ contains no nontrivial $T$-invariant closed subsets. In [12], Putnam proved that if the zero-dimensional system $(X, T)$ is minimal and $X$ has no isolated points, then $\text{sr}(C(X) \times_T Z) = 1$. A key step in his proof is that for every nonempty closed subset $Y$ of $X$, the $C^*$-subalgebra $A_Y$ of $C(X) \times_T Z$ generated by $C(X)$ and $\{Uf: f \in C(X), f(y) = 0 \text{ for all } y \in Y\}$ is an AF-algebra—i.e., $A_y$ is the closure of an increasing sequence of finite-dimensional subalgebras [5]. This result has been generalized to the following:

**Proposition 1.1** [11, Theorem 2.2]. Given any zero-dimensional system $(X, T)$ and a nonempty closed subset $Y$ of $X$, the subalgebra $A_Y$ is AF if and only if $\bigcup_{n \in \mathbb{Z}} T^n(W) = X$ for every clopen subset $W$ containing $Y$.

We will use $D(X, T)$ to denote the set of closed subsets $Y$ of $X$ satisfying the condition in the above proposition. Suppose $Y \in D(X, T)$ is $T$-invariant. Theorem 3.1 gives three conditions equivalent to $\text{sr}(C(X) \times_T Z) = 1$, one of which is that $\text{sr}(C(Y) \times_T Z) = 1$ and every $T$-invariant clopen subset of $Y$ is the intersection of $Y$ and a $T$-invariant clopen subset of $X$. Let $E(X, T)$ be the set of minimal (in the sense of inclusion) elements in $D(X, T)$. Suppose $Y \in E(X, T)$ is $T$-invariant. Let $i$ be the embedding of $A_Y$ into $C(X) \times_T Z$. Then Theorem 3.4 shows that $\text{sr}(C(X) \times_T Z) = 1$ if and only if $i_y: K_0(A_Y) \to K_0(C(X) \times_T Z)$ is an isomorphism. A system $(X, T)$ is said to be indecomposable if $X$ and $\emptyset$ are the only $T$-invariant clopen subsets of $X$. In §4, we prove that if $(X, T)$ is an indecomposable zero-dimensional system, then $\text{sr}(C(X) \times_T Z) = 1$ if and only if the intersection of all $T$-invariant nonempty closed subsets of $X$ is nonempty. We conclude with some remarks and an example in connection with a result of Pimsner [9].

We will use Blackadar [1], Effros [5], and Pedersen [8] for our references on $K$-theory, $AF$-algebras and $C^*$-algebras.

Theorem 4.1 has also been proven by Putnam in a revised version of [13], which we received after this paper had been submitted.
2. Stable rank of crossed products

We start with a result of G. Nagy (Nistor [7, Lemma 3]):

**Lemma 2.1.** Let $0 \to I \to A \to B \to 0$ be an exact sequence of $C^*$-algebras such that $\text{sr}(I) = \text{sr}(B) = 1$. Then $\text{sr}(A) = 1$ if and only if the index morphism $\delta : K_1(B) \to K_0(I)$ is zero.

**Lemma 2.2.** Let $0 \to I \xrightarrow{i} A \xrightarrow{\pi} B \to 0$ be an exact sequence of $C^*$-algebras. Then $\text{sr}(A) = 1$ if and only if $\text{sr}(B) = \text{sr}(I) = 1$ and $\pi_* : K_1(A) \to K_1(B)$ is onto.

**Proof.** From $0 \to I \xrightarrow{i} A \xrightarrow{\pi} B \to 0$, we have a six-term exact sequence of $K$-groups [1]. So the result follows from the exactness at $K_1(B)$:

$$K_1(A) \xrightarrow{\pi_*} K_1(B) \xrightarrow{\delta} K_0(I) \xrightarrow{}$$

and Lemma 2.1. □

So far, most of the results on determining $\text{sr}(A) = 1$ have been done on simple $C^*$-algebras $A$ (e.g. [12, 14]). Given a dynamical system $(X, T)$, the crossed product $C(X) \times_T Z$ is not simple if and only if there exists a nonempty $T$-invariant proper closed subset $Y$ of $X$. By restricting the action of $T$ and the functions in $C(X)$ on $Y$, we have a surjective $C^*$-homomorphism $\pi : C(X) \times_T Z \to C(Y) \times_T Z$. Let $I_Y$ be the kernel of $\pi$; then we have an exact sequence of $C^*$-algebras $0 \to I_Y \to C(X) \times_T Z \to C(Y) \times_T Z \to 0$. Applying Lemma 2.2 to this exact sequence, we have

**Theorem 2.3.** Let $(X, T)$ be a dynamical system and $Y$ a nonempty $T$-invariant closed subset of $X$. Then $\text{sr}(C(X) \times_T Z) = 1$ if and only if

$$\text{sr}(I_Y) = \text{sr}(C(Y) \times_T Z) = 1$$

and $\pi_* : K_1(C(X) \times_T Z) \to K_1(C(Y) \times_T Z)$ is onto.

Given a zero-dimensional system $(X, T)$, let $C(X, Z)$ be the group of integer-valued continuous functions on $X$ and $C^T(X, Z)$ the $T$-invariant functions in $C(X, Z)$. Suppose $Y$ is a nonempty closed subset of $X$. Define $Y : C(X, Z) \to C(Y, Z)$ by $Y(f) = f|_Y$, the restriction of $f$ to $Y$.

**Lemma 2.4.** Let $(X, T)$ be a zero-dimensional dynamical system and $Y$ a nonempty $T$-invariant closed subset of $X$. Then $K_1(C(X) \times_T Z) \simeq C^T(X, Z)$, $K_1(C(Y) \times_T Z) \simeq C^T(Y, Z)$ and the map $\pi_* : K_1(C(X) \times_T Z) \to K_1(C(Y) \times_T Z)$ is given by $Y : C^T(X, Z) \to C^T(Y, Z)$.

**Proof.** We compute $K_1(C(X) \times_T Z)$ by the Pimsner and Voiculescu six-term exact sequence [10]:

$$\begin{array}{cccc}
K_1(C(X)) & \xrightarrow{id_* - T_*} & K_1(C(X)) & \xrightarrow{i_*} & K_1(C(X) \times_T Z) \\
K_0(C(X) \times_T Z) & \xrightarrow{i_*} & K_0(C(X)) & \xrightarrow{id_* - T_*} & K_0(C(X))
\end{array}$$
Since $X$ is zero-dimensional, $K_1(C(X)) = 0$. Hence, the map $K_1(C(X) \times_T Z) \to K_0(C(X))$ is always one-to-one. Also, $K_0(C(X))$ is isomorphic to $C(X, Z)$, the integer-valued continuous functions on $X$. Thus, $K_1(C(X) \times_T Z)$ is isomorphic to the kernel of $\text{id}_* - T_*: K_0(C(X)) \to K_0(C(X))$, which is precisely $C^T(X, Z)$ (see [10] for details on the homomorphisms in the exact sequence). For each $f \in C^T(X, Z)$, there exist integers $n_i$, $1 \leq i \leq k$, and a clopen partition $\{O_i: 1 \leq i \leq k\}$ of $X$ such that each $O_i$ is $T$-invariant and $f = \sum_{i=1}^{k} n_i \chi_{O_i}$, where $\chi_{O_i}$ denotes the characteristic function on $O$. Since all $O_i$ are $T$-invariant, $\sum_{i=1}^{k} n_i \chi_{O_i}$ is a unitary of $C(X) \times_T Z$. An analysis of the connecting homomorphisms in the proof in [10] shows that $\sum_{i=1}^{k} n_i \chi_{O_i} \mapsto \sum_{i=1}^{k} n_i \chi_{O_i}$ gives an isomorphism of $C^T(X, Z)$ and $K_1(C(X) \times_T Z)$.

Similarly, $K_1(C(Y) \times_T Z) \cong C^T(Y, Z)$ and the result follows.

**Remark 2.5.** Under the conditions in Lemma 2.4, we note that the map $\Phi_Y: C^T(X, Z) \to C^T(Y, Z)$ is onto if and only if for every $T$-invariant clopen subset $Q$ of $Y$, there exists a $T$-invariant clopen subset $O$ of $X$ such that $Q = O \cap Y$.

**Corollary 2.6.** Suppose $(X, T)$ is a zero-dimensional system with no nontrivial $T$-invariant clopen subsets. If $X$ contains two disjoint nonempty $T$-invariant closed subsets, then $\text{sr}(C(X) \times_T Z) = 2$.

**Example 2.7.** Let $(X, T)$ be a zero-dimensional system which contains a point with dense orbit and two periodic points $x_1, x_2$ with disjoint orbits. Then the conditions in Corollary 2.6 are satisfied and $\text{sr}(C(X) \times_T Z) = 2$. Hence, for most subshifts and all nontrivial irreducible Markov shifts [4] $(X, T)$, $\text{sr}(C(X) \times_T Z) = 2$.

### 3. Subsets in $D(X, T)$ and $E(X, T)$

Throughout this section, $(X, T)$ will denote a zero-dimensional dynamical system. For each nonempty closed subset $Y$ of $X$, $A_Y$ is the subalgebra of $C(X) \times_T Z$ generated by $C(X)$ and $\{Uf: f \in C(X), f(y) = 0 \text{ for all } y \in Y\}$. Let $D(X, T)$ be the set of closed subsets $Y$ of $X$ such that $\bigcup_{n \in Z} T^n(W) = X$ for every clopen subset $W$ containing $Y$. By Proposition 1.1, $A_Y$ is an AF subalgebra if and only if $Y \subseteq D(X, T)$. Let $E(X, T)$ be the set of minimal (in the sense of inclusion) elements in $D(X, T)$ [11].

**Theorem 3.1.** Let $Y$ be a $T$-invariant subset in $D(X, T)$ and $\pi: C(X) \times_T Z \to C(Y) \times_T Z$ as defined in §2. Then the following are equivalent:

1. $\text{sr}(C(X) \times_T Z) = 1$.
2. $\text{sr}(C(Y) \times_T Z) = 1$ and $\pi_*: K_1(C(X) \times_T Z) \to K_1(C(Y) \times_T Z)$ is an isomorphism.
3. $\text{sr}(C(Y) \times_T Z) = 1$ and $\Phi_Y: C^T(X, Z) \to C^T(Y, Z)$ is an isomorphism.
(4) \( \text{sr}(C(Y) \times_T Z) = 1 \) and for each \( T \)-invariant clopen subset \( Q \) of \( Y \) there exists a \( T \)-invariant clopen subset \( O \) of \( X \) such that \( Q = O \cap Y \).

**Proof.** Let \( I_Y \) be the kernel of \( \pi \), i.e., \( I_Y \) is the ideal of \( C(X) \times_T Z \) generated by functions in \( C(X) \) vanishing in \( Y \). Since \( Y \) is \( T \)-invariant, \( I_Y \) is an ideal of the \( AF \) subalgebra \( A_Y \). Thus, \( I_Y \) is also \( AF \). So we have \( \text{sr}(I_Y) = 1 \) and \( K_1(I_Y) = 0 \). Hence, \( \pi_* \) is one-to-one and the result follows from Theorem 2.3, Lemma 2.4, and Remark 2.5.

**Remark 3.2.** In the above theorem, since \( \pi_* \) is always one-to-one, the “isomorphism” conditions in (2) and (3) can be replaced by “onto”.

Before proving the next theorem, we need the following generalization of a result of Putnam [13, Theorem 4.1]:

**Proposition 3.3.** Let \( Y \in D(X,T) \). There is an exact sequence
\[
0 \to C^T(X,Z) \xrightarrow{\Phi_Y} C(Y,Z) \xrightarrow{\nu} K_0(A_Y) \xrightarrow{i} K_0(C(X) \times_T Z) \to 0.
\]

Here \( i \) is the embedding of \( A_Y \) into \( C(X) \times_T Z \). For our application, the definition of \( \nu \) is not important. We include it here just for completeness: Given \( f \in C(Y,Z) \), we choose \( g \in C(X,Z) \) such that \( g|_Y = f \). Let \( i_1 \) be the embedding of \( C(X) \) into \( A_Y \) and \( i_* : K_0(C(X)) \to K_0(A_Y) \). Identifying \( K_0(C(X)) \) with \( C(X,Z) \), we put \( \psi(f) = i_1(g - g \circ T) \). This definition is due to Putnam in [13], where he proved the result for minimal systems \((X,T)\).

But, the proof for the general case is essentially the same.

**Theorem 3.4.** Let \( Y \) be a \( T \)-invariant subset in \( E(X,T) \). The following conditions are equivalent:

1. \( \text{sr}(C(X) \times_T Z) = 1 \).
2. \( \pi_* : K_1(C(X) \times_T Z) \to K_1(C(Y) \times_T Z) \) is an isomorphism.
3. \( \Phi_Y : C^T(X,Z) \to C(Y,Z) \) is an isomorphism.
4. \( i_* : K_0(A_Y) \to K_0(C(X) \times_T Z) \) is an isomorphism.
5. For each clopen subset \( Q \) of \( Y \) there exists a \( T \)-invariant clopen subset \( O \) of \( X \) such that \( Q = O \cap Y \).

**Proof.** First we note that for any \( Y \in D(X,T) \), conditions (3), (4) and (5) are always equivalent by Proposition 3.3.

Let \( Y \in E(X,T) \) be \( T \)-invariant. We are going to prove that \( T(y) = y \) for all \( y \in Y \). Suppose the contrary that \( T(y) \neq y \) for some \( y \in Y \). Then we can choose a clopen subset \( O \) of \( X \) containing \( y \) such that \( O \cap T(O) = \emptyset \). So, we have that \( Y \setminus O \) is a proper closed subset of \( Y \). Let \( W \) be a clopen subset of \( X \) containing \( Y \setminus O \); we have
\[
T^{-1}(W) \supseteq T^{-1}(Y \setminus O) \supseteq T^{-1}(Y \cap T(O)) \supseteq Y \cap O
\]
\[
\Rightarrow W \cup T^{-1}(W) \supseteq Y
\]
\[
\Rightarrow \bigcup_{n \in \mathbb{Z}} T^n(W) = \bigcup_{n \in \mathbb{Z}} T^n(W \cup T^{-1}(W)) = X.
\]

Thus, \( Y \setminus O \in D(X,T) \), contradicting \( Y \in E(X,T) \).
Since the action of $T$ on $Y$ is the identity, $C(Y) \times_T Z$ is isomorphic to the $C^*$-tensor product $C(Y) \otimes C(S)$ [8], where $S$ is the unit circle. Since $Y$ is zero-dimensional, $sr(C(Y) \otimes C(S)) = 1$. So the result follows from Theorem 3.1 because $C^T(Y, Z) = C(Y, Z)$. □

Remark 3.5. If $Y \in E(X, T)$ consists of a single fixed point, then condition (5) and hence all conditions in Theorem 3.4 are obviously satisfied. In Corollary 4.2, we will give a partial converse of this result. Here, we give a class of systems satisfying this condition:

Let $T$ be a continuous strictly increasing function on the unit interval $[0, 1]$ with $T(0) = 0$, $T(1) = 1$ and $T(x) \neq x$ for $0 < x < 1$. Choose a countable $T$-invariant dense subset $S$ of the open interval $(0, 1)$. For each $0 < s < t < 1$, let $x_{[s, t)}$ be the characteristic function on the interval $[s, t)$. Let $A$ be the commutative $C^*$-algebra generated by $\{x_{[s, t)} : s, t \in S\}$ and the constant function 1. Then $A$ is isomorphic to $C(X)$ for a zero-dimensional space $X$ which contains the interval $[0, 1)$ and the action of $T$ extends to $X$. One checks that 0 is a fixed point and $\{0\} \in E(X, T)$.

The above construction is similar to the one of Cuntz [3, Example 2.5]. Similar examples can also be constructed on higher-dimensional analogues of the unit interval.

4. INDECOMPOSABLE SYSTEMS

Given a system $(X, T)$, if $X$ can be decomposed into two disjoint nonempty $T$-invariant closed subsets $X_1$ and $X_2$, then $C(X) \times_T Z$ is isomorphic to the direct sum $\bigoplus_{i=1}^2 C(X_i) \times_T Z$. Hence, $sr(C(X) \times_T Z) = 1$ if and only if $sr(C(X_i) \times_T Z) = 1$, for $i = 1, 2$. $(X, T)$ is called indecomposable if no such decomposition exists, i.e., the only $T$-invariant clopen subsets of $X$ are $X$ and $\emptyset$.

Theorem 4.1. Let $(X, T)$ be an indecomposable zero-dimensional dynamical system. Then $sr(C(X) \times_T Z) = 1$ if and only if the intersection of all nonempty $T$-invariant closed subsets of $X$ is nonempty.

Proof. Let $Y$ be the intersection of all nonempty $T$-invariant closed subsets of $X$.

Suppose $sr(C(X) \times_T Z) = 1$. Since $(X, T)$ is indecomposable, by Corollary 2.6, the intersection of any two nonempty $T$-invariant closed subsets of $X$ is nonempty. Thus, by the finite intersection property, $Y$ is nonempty.

Conversely, suppose $Y$ is nonempty. Clearly, $Y$ is a $T$-invariant closed subset of $X$. We are going to show that (1) $Y \in D(X, T)$ and (2) the action of $T$ on $Y$ is minimal. Then the result will follow from (4) in Theorem 3.1 because $sr(C(Y) \times_T Z) = 1$ for a minimal system $(Y, T)$ (Putnam [13]).

To prove (1), let $W$ be a clopen subset containing $Y$. Then $X \setminus \bigcup_{n \in \mathbb{Z}} T^n(W)$ is a $T$-invariant closed subset disjoint from $Y$. Thus $X \setminus \bigcup_{n \in \mathbb{Z}} T^n(W) = \emptyset$ and $\bigcup_{n \in \mathbb{Z}} T^n(W) = X$. Hence, $Y \in D(X, T)$.
To prove (2), for every \( y \in Y \), the orbit closure of \( y \) is a \( T \)-invariant closed subset of \( Y \) and hence is equal to \( Y \).

**Corollary 4.2.** Suppose \((X, T)\) is an indecomposable zero-dimensional system such that \( \text{sr}(C(X) \times_T Z) = 1 \). Then every \( T \)-invariant \( Y \) in \( E(X, T) \) consists of a single point.

**Proof.** From the proof of Theorem 3.4, we have that \( T(y) = y \) for every \( y \in Y \). Thus, \( Y \) is nonempty only when it consists of a single point.

**Remark 4.3.** Let \((X, T)\) be a (not necessarily zero-dimensional) dynamical system. A point \( x \in X \) is said to be pseudo-nonwandering (Pimsner [9]) if for every open cover \( \{O_i\}_{i=1}^m \) of \( X \) and \( O_i \) containing \( x \), there exist \( O_k, 2 \leq k \leq m \), such that \( O_i \cap T^{-1}(O_{j+1}) \neq \emptyset \) for \( 2 \leq j < m \) and \( O_{j+1} \cap T^{-1}(O_i) \neq \emptyset \).

Let \( X(T) \) be the set of all pseudo-nonwandering points in \( X \). Pimsner [9] proved that the following three conditions are equivalent: (1) \( X(T) = X \), (2) \( C(X) \times_T Z \) contains no nonunitary isometry, and (3) there is a unital imbedding of \( C(X) \times_T Z \) into an AF algebra. Since a \( C^* \)-algebra with stable rank 1 cannot contain any nonunitary isometry, \( X(T) = X \) is a necessary condition for \( \text{sr}(C(X) \times_T Z) = 1 \) (this connection is communicated to us by Putnam). The following example shows that the condition is not sufficient.

**Example 3.7.** Let \( X = \mathbb{Z} \cup \{\infty, -\infty\} \) be the two-point compactification of the integers. Define a homeomorphism \( T: X \to X \) by

\[
T(\infty) = \infty, \quad T(-\infty) = -\infty, \\
T(x) = \begin{cases} 
  x + 2, & \text{if } x \in \mathbb{Z}, \text{ even,} \\
  x - 2, & \text{if } x \in \mathbb{Z}, \text{ odd.}
\end{cases}
\]

It is straightforward to check that \( X(T) = X \) and \( Y = \{\infty, -\infty\} \) is a \( T \)-invariant subset in \( E(X, T) \). Hence, \( \text{sr}(C(X) \times_T Z) \neq 1 \) by Corollary 4.2.

We note from Example 2.7 that any nontrivial irreducible Markov shift can also serve this purpose.

**References**


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