On $C$-det spectral and $C$-det-convex Matrices

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Denote by $U_n$, the group of the $n \times n$ unitary matrices. Let $A$ be a complex $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and $C = \text{diag}(c_1, \ldots, c_n) \in \mathbb{C}^n$. The set

$$\Delta_n(A) = \{\det(C + UAU^*: U \in U_n)\}$$

of complex numbers will be called the determinantal range of $A$. Let $P_n(A)$ be the convex hull of $\{\sum_{\sigma \in S_n} \sigma(C + UAU^*): C \in \mathbb{C}^n\}$. The matrices for which $\Delta_n(A) = \delta_n(A)$ and $P_n(A) = A_n(A)$ are investigated. Some new analogies between $A_n(A)$ and the well-known $c$-numerical range of $A$ are found. Also, the interesting interplay between the geometric properties of $A_n(A)$ and the algebraic properties of $A$ is re-examined and strengthened by the results obtained here. A necessary and sufficient condition for the convexity of $A_n(A)$ in terms of the eigenvalues of $A$ and $C$ for $3 \times 3$ normal matrices is given. A sufficient condition for the convexity of $A_n(A)$ when $A$ and $C$ are, respectively, a $3 \times 3$ matrix and a $3 \times 3$ normal matrix is presented.

1. INTRODUCTION

Denote by $U_n$ the group of $n \times n$ unitary matrices. Given an $n \times n$ complex matrix $A$ with eigenvalues $\lambda_j, 1 \leq j \leq n$, and a complex row vector $c = (\gamma_1, \ldots, \gamma_k)$, let $C = \text{diag}(\gamma_1, \ldots, \gamma_k)$. Define the complex set
of points
\[ \Delta_c(A) := \{ \det(C + UAU^*) : U \in \mathcal{U}_n \}, \]
which will be called the c-determinantal range of \( A \). This set is compact and connected. However, as it has been shown in [4], it may not be simply connected.

Let
\[ P_c(A) = \text{Co} \left\{ \prod_{j=1}^{n} (v_j + z_{\sigma(j)}) : \sigma \in S_n \right\}, \]
where Co\{ \cdot \} is the convex hull of the set \{ \cdot \} and \( S_n \) is the symmetric group of degree \( n \). The set \( P_c(A) \) will be called the c-det-eigenpolygon of \( A \). Define the c-det-spectral radius of \( A \)
\[ \delta_c(A) = \max \{ \|z\| : z \in P_c(A) \}, \]
and the c-determinantal radius
\[ d_c(A) = \max \{ \|z\| : z \in \Delta_c(A) \}. \]
The points \( z_\sigma = \prod_{j=1}^{n} (v_j + z_{\sigma(j)}) \), \( \sigma \in S_n \), will be called \( \sigma \)-points. Since the \( n! \) \( \sigma \)-points all belong to \( \Delta_c(A) \), the following relation holds
\[ \delta_c(A) \leq d_c(A). \]
We shall call \( A \) c-det-convex if
\[ P_c(A) = \Delta_c(A) \]
and c-det-spectral if
\[ d_c(A) = \delta_c(A). \]
The main purpose of this note is the characterization of the c-det-convex and the c-det-spectral matrices.

In particular, a necessary and sufficient condition for the convexity of \( \Delta_c(A) \) in terms of the eigenvalues of \( A \) and \( C \) for \( 3 \times 3 \) normal matrices is presented.

Also, an improvement of Theorem 2 of [3] is given (Theorem 2.2). We emphasize the curious interplay between the geometric properties of \( \Delta_c(A) \) and the algebraic properties of \( A \) which underlies our results. Moreover, we observe that certain analogies can be found between our situations and the corresponding ones concerning the set \( W_c(A) = \{ \text{tr}(AUU^*) : U \in \mathcal{U}_n \} \), called the c-numerical range of \( A \). These
analogies, which have already been pointed out in previous works ([2], [3], [4]) are reexploited and remarked throughout this note. In particular, we observe the analogy between some of our results and parallel results for $W_f(A)$ due to Au-Yeung, Poon [1] and Li, Tam, Tsing [6], [7].

2. C-DET-SPECTRAL MATRICES

We denote by $M_n(C)$ the linear space of $n \times n$ complex matrices and by $s(A)$, the spectrum of $A$. Let $z$ belong to the boundary $\partial \Delta_c(A)$ of $\Delta_c(A)$. If, in the neighborhood of $z$, $\Delta_c(A)$ is contained in an angle with vertex at $z$ and measuring less than $\pi$, then $z$ is called a corner.

LEMMA 2.1 (see [2, Theorem 2]) If $0 \neq z \in \partial \Delta_c(A)$ is a corner, then $z$ is a $\sigma$-point.

LEMMA 2.2 (see [3, Corollary 3]) Let

$$A = \begin{bmatrix}
\lambda_1 & a_{12} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots & \vdots \\
a_{n1} & \cdots & \lambda_n
\end{bmatrix} \in M_n(C)$$

and $c = (\gamma_1, \ldots, \gamma_n) \in C^n$. If $\prod_{j=1}^n (\gamma_j + z_j) \in \partial \Delta_c(A)$, then $a_{ij} = 0$ for $\gamma_i \neq \gamma_j$ ($i \neq j$).

In the rest of this section, for $c = (\gamma_1, \ldots, \gamma_n) \in C^n$, we always assume,

$$c_1 = \gamma_1 = \cdots = \gamma_{n_1},$$
$$c_2 = \gamma_{n_1} + 1 = \cdots = \gamma_{n_1 + n_2},$$
$$\vdots$$
$$c_k = \gamma_{n_1 + \cdots + n_{k-1} + 1} = \cdots = \gamma_{n_1 + \cdots + n_k},$$

where $n_1 + \cdots + n_k = n$ and $c_1, \ldots, c_k$ are distinct.

THEOREM 3.1 Let $c = (\gamma_1, \ldots, \gamma_n) \in C^n$ and $A \in M_n(C)$. Then $A$ is $c$-det-spectral if and only if $A$ is unitarily similar to $A_1 \oplus \cdots \oplus A_k$, where $A_j \in M_{n_j}(C), j = 1, \ldots, k$ with $d_i(A) = \prod_{j=1}^{n_j} \det(c_j I_{n_j} + A_j)$.

Proof ($\Rightarrow$) Let $\delta_c(A) = \prod_{j=1}^n (\gamma_j + z_{n(j)}), \sigma \in S_n$. By Schur's Triangularization Theorem, we can write $A = UDU^*$, where $D$ is a diagonal matrix with $d_i(A)$ on its diagonal. Then $A$ is unitarily similar to $A_1 \oplus \cdots \oplus A_k$, where $A_j = U_j D_j U_j^*$, $j = 1, \ldots, k$, and $D_j$ is a diagonal matrix with $d_i(A)$ on its diagonal, $i = 1, \ldots, n_j$. Therefore, $A$ is $c$-det-spectral.

Proof ($\Leftarrow$) Let $\delta_c(A) = \prod_{j=1}^n (\gamma_j + z_{n(j)}), \sigma \in S_n$. By Schur's Triangularization Theorem, we can write $A = UDU^*$, where $D$ is a diagonal matrix with $d_i(A)$ on its diagonal. Then $A$ is unitarily similar to $A_1 \oplus \cdots \oplus A_k$, where $A_j = U_j D_j U_j^*$, $j = 1, \ldots, k$, and $D_j$ is a diagonal matrix with $d_i(A)$ on its diagonal, $i = 1, \ldots, n_j$. Therefore, $A$ is $c$-det-spectral.
gularization Lemma, \( A \) is unitarily similar to
\[
\begin{bmatrix}
 z_{\sigma(1)} & a_{ij} \\
 & \ddots \\
 0 & & z_{\sigma(n)}
\end{bmatrix}.
\]
For simplicity, we assume that \( \sigma \) is the identity permutation. Since
\[
d_\sigma(A) = \delta_\sigma(A), \prod_{j=1}^{n} \left( \gamma_j + z_{\sigma(j)} \right) \in \delta A_\sigma(A).
\]
By Lemma 2.2, \( a_{ij} = 0 \) for \( j \neq i \) (\( i < j \)), and the result follows.

(\( \Rightarrow \)) By the assumptions in the hypothesis, \( A \) is unitarily similar to
\( A_1 \oplus \cdots \oplus A_k \). Since the eigenvalues of \( A_j, j = 1, \ldots, k, \) are eigenvalues of
\( A \) and
\[
d_\sigma(A) = \prod_{j=1}^{k} \det(c_j I_{n_j} + A_j),
\]
we have
\[
d_\sigma(A) = \prod_{j=1}^{k} \left( \gamma_j + z_{\sigma(j)} \right)
\]
for some \( \sigma \in S_n \). Thus, \( d_\sigma(A) \leq \delta_\sigma(A) \). As \( \delta_\sigma(A) \leq d_\sigma(A) \), the result follows.

The following two corollaries are obvious consequences of this theorem.

**Corollary 2.1** Suppose \( c = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n \) is such that the \( \gamma_j \)'s are
distinct and \( A \in M_\sigma(\mathbb{C}) \). If \( A \) is \( c \)-det-spectral, then \( A \) is normal.

We recall that \( A \) is said to be unitarily irreducible if it is not unitarily
similar to the direct sum \( A_1 \oplus \cdots \oplus A_k \) of two matrices \( A_1, A_2 \).

**Corollary 2.2** Suppose \( A \in M_\sigma(\mathbb{C}) \) is unitarily irreducible. Then the
following conditions are equivalent:

(i) \( A \) is \( c \)-det-spectral;

(ii) \( C \) is a scalar matrix;

(iii) \( A \) is \( c \)-det-convex.

The following theorem is analogous to Theorem 3.1 of [7].

**Theorem 2.2** Let \( c = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n \) and \( A \in M_\sigma(\mathbb{C}) \). If there exists
\( \sigma \in S_n \) such that \( z_\sigma = \prod_{j=1}^{n} \left( \gamma_j + z_{\sigma(j)} \right) \in \delta A_\sigma(A) \), then \( A \) is unitarily similar to
\( A_1 \oplus \cdots \oplus A_k \), where \( A_i \in M_\sigma(\mathbb{C}) \), \( 1 \leq i \leq k \), is an upper triangular
matrix. Moreover, each \( A_j \) is of the form \( \text{diag}(z_{j,1}^{(1)}, \ldots, z_{j,n}^{(1)}) \oplus B_j \), where
By Schur's Triangularization Lemma, $A$ is unitarily similar
to an upper triangular matrix. Since $\Delta_i(A)$ is unitarily invariant, the
first assertion is an obvious consequence of Lemma 2.2. Moreover, we
may assume that each $A_i$ is of the form

\[
A_i = \begin{bmatrix}
    a(i,j) \\
    a(j,i) \\
    \vdots \\
    0 \\
    a(n,j)
\end{bmatrix}
\]

with $a(i,0), a(n,0) \in s(A_i)$ and $a(n,1), \ldots, a(n,i) \notin \bigcup_{i \neq j} s(A_j)$. We are
going to prove that $a(i,j) = 0$ for all $i < j$. For simplicity, we assume that $\sigma$ is the identity, $A_1 \oplus \cdots \oplus A_k = (a_{ij})$
and $(a_1, \ldots, a_k) = (a(1,1), \ldots, a(n,k), \ldots, a(n,k)).$

Since $1 \leq p \leq m_i$, there exists $j \neq i$ such that $a(i,j) = a(j,i)$ for some $1 \leq r \leq n_j$. Let $s = n_1 + \cdots + n_{i-1} + p$, $r = n_1 + \cdots + n_{j-1} + r$ and
c' = $(\gamma_1', \ldots, \gamma_n')$ be the vector obtained from $c$ by switching $\gamma_j$ and $\gamma_i$. We have

\[
\prod_{j=1}^{n} (\gamma_j + x_j) = \prod_{j=1}^{n} (\gamma_j + x_j) \in \Delta_\sigma(A) = \Delta_\sigma(A).
\]

For each $p < q \leq n$, let $u = n_1 + \cdots + n_{i-1} + p$. We have $s < u \leq n_1 + \cdots + n_i$. Thus, by Lemma 2.2, $a(i,j) = a(u,v) = 0$ because $\gamma_j = \gamma_i = \gamma_j = \gamma_i$.

**COROLLARY 2.3** Let $c = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$ and $A$ be a $c$-det-spectral
matrix. If $C$ is an eigenvalue of $A$ with multiplicity $p$ such that $p > \max_{1 \leq i \leq k} n_i$, then $A$ is unitarily similar to $xI_p \oplus B$, where $B \in M_{n-p}(\mathbb{C})$.

**COROLLARY 2.4** Let $C = I_m \oplus 0_{n-m}$, and $A \in M_n(\mathbb{C})$. If

\[
d_i(A) = \prod_{j=1}^{m} (1 + x_{\sigma(j)}) \prod_{j=m+1}^{n} x_{\sigma(j)}
\]

for some $\sigma \in S_n$, then $A$ is unitarily similar to $A_1 \oplus A_2$, where $A_1 \in M_m(\mathbb{C})$ and

\[
A_1 = \text{diag}(\lambda_1, \ldots, \lambda_j) \oplus B_1, \quad A_2 = \text{diag}(\mu_1, \ldots, \mu_k) \oplus B_2.
\]
where \( s(B_2) \cap s(B_1) = \emptyset \), \( s(B_1) \cap s(A_2) = \emptyset \) and
\[
\{\lambda_1, \ldots, \lambda_s\} = \{\mu_1, \ldots, \mu_s\} = s(A_1) \cap s(A_2).
\]

Remark 1 Compare this corollary with Theorem 4.8 of [6].

3. C-DET-CONVEX MATRICES

**Theorem 3.1** (cf. [7, Theorem 4.1]) Let \( c = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n \); \( A \in M_n(\mathbb{C}) \) be such that the origin is not a vertex of \( P_c(A) \). Then \( A \) is c-det-convex if and only if \( \partial \Delta_c(A) \) is a convex polygon in \( \mathbb{C} \).

**Proof** (\( \Rightarrow \)) Clear.

(\( \Leftarrow \)) The inclusion \( P_c(A) \subseteq \Delta_c(A) \) is a trivial consequence of the hypothesis. On the other hand, every vertex of \( \Delta_c(A) \) is a corner and thus, by Lemma 2.1, is a \( \sigma \)-point. Therefore, every vertex of \( \Delta_c(A) \) is in \( P_c(A) \). Hence, \( \Delta_c(A) \subseteq P_c(A) \). This proves the theorem. \( \square \)

**Theorem 3.2** Let \( c = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n \) and \( A \in M_n(\mathbb{C}) \) with \( s(A) = \{x_1, \ldots, x_n\} \). The following are equivalent:

(i) \( \Delta_c(A) \) is a point of \( \mathbb{C} \setminus \{0\} \);

(ii) \( P_c(A) \) is a point of \( \mathbb{C} \setminus \{0\} \) and \( A \) is c-det-convex;

(iii) either \( A \) or \( C \) is a scalar matrix and \( \prod_{i=1}^n (\gamma_i + x_i) \neq 0 \).

**Proof** (iii) \( \Rightarrow \) (ii) Trivial.

(i) \( \Rightarrow \) (iii) Suppose (i) is satisfied and \( \gamma_i \neq \gamma_j \) for some \( 1 \leq i < j \leq n \). We will first prove that \( A \) is normal. For simplicity, assume \( \gamma_1 \neq \gamma_2 \) and \( A = (a_{ij}) \) is upper triangular. Then for every \( 1 \leq i < j \leq n \), choose \( \sigma \in S_n \) such that \( \sigma(i) = 1, \sigma(j) = 2 \). Let \( c' = (\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)}) \), then we have
\[
\prod_{k=1}^n (\gamma_{\sigma(k)} + x_k) \in \Delta_c(A) = \Delta_{c'}(A) = \partial \Delta_{c'}(A)
\]
and \( \gamma_{\sigma(i)} \neq \gamma_{\sigma(j)} \). Thus, by Lemma 2.2, \( a_{ij} = 0 \). Hence, \( A \) is normal. Since \( \gamma_1 \neq \gamma_2 \) and \( \prod_{i=1}^n (\gamma_i + x_{\sigma(i)}) \) is non-zero and independent of \( \sigma \in S_n \), it follows that \( x_1 = \cdots = x_n \) and \( A \) is a scalar matrix. \( \square \)

Remark 2 If \( A = \text{diag}(1, 1, -1) \) and \( c = (1, -1, -1) \), then direct computation shows that \( \Delta_c(A) = \{0\} \) but neither \( A \) nor \( C \) is a scalar matrix.

Suppose now that \( c = (\gamma_1, \gamma_2, \gamma_3) \) and \( A = \text{diag}(x_1, x_2, x_3) \). In [5], an example has been given such that \( \Delta_c(A) \) is not convex. Our next theorem
provides a necessary and sufficient condition for \( \Delta_\nu(A) \) to be convex. This theorem and its proof is parallel to similar results in [1] on the \( \nu \)-numerical range \( W_\nu(A) \).

Recall that an \( n \times n \) matrix \( S = (s_{ij}) \) is said to be orthostochastic (o.s.) if there exists a unitary matrix \( (u_{ij}) \) such that \( s_{ij} = |u_{ij}|^2 \). Let \( O_n \) be the set of all \( n \times n \) o.s. matrices.

Given \( u = (u_{ij}) \in \mathbb{U}_n \), we have [5]

\[
\det(C + UAU^*) = \gamma_1 \gamma_2 \gamma_3 + z_1 z_2 \gamma_3 + c S \bar{a}^T + \bar{c} a^T
\]

where \( a = (x_2, x_3, x_4), \bar{a} = (x_2 x_3, x_3 x_4, x_4 x_2), \bar{c} = (x_2 x_3, x_3 x_4, x_4 x_2) \) and \( S = (|u_{ij}|^2) \). Thus, putting \( D(S) = \gamma_1 \gamma_2 \gamma_3 + z_1 z_2 \gamma_3 + c S \bar{a}^T + \bar{c} a^T \) for \( S \in O_n \), we have \( \Delta_\nu(A) = \{ D(S) : S \in O_n \} \). Since \( O_n \) is star shaped [1] with respect to the matrix

\[
S_0 = \begin{bmatrix}
1/3 & 1/3 & 1/3 \\
1/3 & 1/3 & 1/3 \\
1/3 & 1/3 & 1/3
\end{bmatrix}
\]

and \( D(tS_1 + (1 - t)S_2) = tD(S_1) + (1 - t)D(S_2) \) for \( 0 \leq t \leq 1, S_1, S_2 \in O_3 \), we have

**Lemma 3.1** \( \Delta_\nu(A) \) is star shaped with respect to \( z_0 = D(S_0) \), i.e., for every \( z \in \Delta_\nu(A), D_0 + (1 - t)z \in \Delta_\nu(A) \) for all \( 0 \leq t \leq 1. \)

Given a permutation \( \sigma \in S_3 \), we note that \( \prod_{j=1}^3 (\gamma_j + x_{\sigma(j)}) = D(P_{\sigma}) \) where \( P_\sigma \) is the permutation matrix \( (P_{ij}) \) with \( P_{i\sigma(i)} = 1, 1 \leq i \leq 3 \) and 0 elsewhere. Define the sets

\[
V_+ = \left\{ \prod_{j=1}^3 (\gamma_j + x_{\sigma(j)}): \sigma \text{ is an even permutation} \right\}
\]

and

\[
V_- = \left\{ \prod_{j=1}^3 (\gamma_j + x_{\sigma(j)}): \sigma \text{ is an odd permutation} \right\}.
\]

Since every \( S \in O_n \) is a convex combination of \( P_\sigma, \sigma \in S_3, \Delta_\nu(A) \subseteq \text{co}(V_+ \cup V_-) = P(A) \).

For any two distinct complex numbers \( x, y \), let \( L(x, y) \) denote the line on the complex plane passing \( x \) and \( y \).

With the above description of \( \Delta_\nu(A) \), the proof of the following
theorem is identical to Theorem 6 in [1]. We include it here for completeness.

**Theorem 3.3** \( \Delta(A) \) is not convex if and only if there exist distinct \( x \) and \( y \) in \( V_+ \) (or in \( V_- \)) such that all points in \( V_- \) (or \( V_+ \) respectively) lie on one side (the open half plane) of \( L(x, y) \).

**Proof** We note that for any \( x \in V_+ \) and \( y \in V_- \), the line segment joining \( x \) and \( y \) lies in \( \Delta(A) \). Since \( \Delta(A) \) lies in \( P_c(A) \) and is star shaped, \( \Delta(A) \) is convex if and only if the boundary of \( P_c(A) \) lies in \( \Delta(A) \).

If the condition in the theorem is not satisfied, then the adjacent vertices of \( P_c(A) \) lie in \( V_+ \) and \( V_- \) alternately. Thus, \( \Delta(A) \) is convex.

Conversely, if there exist distinct points \( x = D(P_{s_1}), y = D(P_{s_2}) \) in \( V_+ \) (or in \( V_- \)) such that all points in \( V_- \) (or in \( V_+ \) respectively) lie on one side of \( L(x, y) \), then from

\[
 z_0 = \frac{1}{3} \sum_{z \in V_-} z = \frac{1}{3} \sum_{z \in V_+} z,
\]

the third point in \( V_- \) (or \( V_+ \)) also lies on this side. Thus, if \( 0 < t < 1 \) and \( tx + (1-t)y \) is equal to a convex combination \( t \sigma_1 + (1-t) \sigma_2 \) of points in \( V_+ \cup V_- \), we have \( t_1 = t, (1-t) = t_2 \). Hence, if \( tx + (1-t)y \notin \Delta(A) \), then we have \( tx + (1-t)y = D(S) \) for some \( S \in 0 \). By writing \( S \) as a convex combination of \( \sigma_1, \sigma_2 \in S_3 \), we have \( S = tP_{s_1} + (1-t)P_{s_2} \). However, one can show directly [1, Theorem 5] that for any two distinct even (or odd) permutations \( \sigma_1, \sigma_2 \), \( tP_{s_1} + (1-t)P_{s_2} \) is not o.s. for all \( 0 < t < 1 \). This contradiction shows that \( tx + (1-t)y \notin \Delta(A) \) for all \( 0 < t < 1 \) and \( \Delta(A) \) is not convex.

**Corollary 3.1** Let \( A \) be a 3 × 3 complex matrix and \( c = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{C}^3 \).

If:

(i) there exist at least two \( \sigma \)-points in \( \partial \Delta(A) \) and

(ii) for any distinct \( x \) and \( y \) in \( V_+ \) (or in \( V_- \)) all the points in \( V_- \) (or \( V_+ \) respectively) do not lie on the same half plane defined by \( L(x, y) \),

then \( \Delta(A) \) is convex and \( \Delta(A) = P_c(A) \).

**Proof** By property 7 of [4], the hypothesis 1 implies that \( A \) is normal. Then the previous theorem applies.
References


