SOLUTIONS FOR QUIZ 3

Problem 1.

$$Y(s) = \frac{e^{-s}+2}{s^2-2s+2} = \frac{e^{-s}}{(s-1)^2+1} + \frac{2}{(s-1)^2+1}.$$

Note that

$$\mathcal{L}^{-1}\{\frac{2}{(s-1)^2+1}\} = 2e^t sin(t).$$

Then

$$\mathcal{L}^{-1}\{e^{-s}\frac{2}{(s-1)^2+1}\} = H(t-1)e^{t-1}sin(t-1).$$

Therefore

$$y(t) = H(t-1)e^{t-1}sin(t-1) + 2e^t sin(t).$$

Problem 2. We represent a solution $y(t)$ as a sum of two functions $y(t) = w(t) + v(t)$. Here

$$w'' + w = H(t-2),\ w(0) = w'(0) = 0, \quad v'' + v = t\ v(0) = v'(0) = 0.$$

In order to find a function $v(t)$ we will use the method undetermined coefficients

$$v_p(t) = q_0 + q_1 t, v_p'' = 0$$

Hence $v_p(t) = t$ and

$$v(t) = C_1 cos(t) + C_2 sin(t) + t$$

From the initial conditions we have

$$C_1 = 0, C_2 = -1.$$

Finally

$$v(t) = -sin(t) + t$$

In order to find a function $w(t)$ we use a method of the Laplace transform

$$W(s) = \frac{e^{-2s}}{s(s^2+1)} = e^{-2s}(\frac{1}{s} - \frac{s}{s^2+1})$$

Since $\mathcal{L}^{-1}\{(\frac{1}{s} - \frac{s}{s^2+1})\} = 1 - cos(t)$. So

$$\mathcal{L}W^{-1} = H(t-2)(1 - cos(t-2)).$$

Finally we have

$$y(t) = -sin(t) + t + H(t-2)(1 - cos(t-2)).$$

Problem 3.

$$F(s) = \frac{5}{s^2(s^2+s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+s+1} = \frac{(C+B)s^3 + (B+D+A)s^2 + (A+B)s + A}{s^2(s^2+s+1)}.$$

Hence

$$C+B=0, \quad B+D+A=0, \quad A+B=0, \quad A=5$$

Finally we have

$$A=5, B=-5, D=0, C=5.$$

and

$$F(s) = \frac{5}{s^2(s^2+s+1)} = \frac{5}{s^2} - \frac{5}{s} + \frac{5s}{s^2+s+1} =$$

$$\frac{5}{s^2} - \frac{5}{s} + 5(\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} - \frac{1}{2}\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}).$$

Then

$$\mathcal{L}^{-1}(F(s)) = 5t - 5 + 5(e^{-\frac{1}{2}t}cos(\sqrt{\frac{3}{4}}t) - \frac{1}{2}e^{-\frac{1}{2}t}sin(\sqrt{\frac{3}{4}}t)).$$

Problem 4. Note that $sin(t)\delta(t-\pi) = sin(\pi)\delta(t-\pi) = 0$ Therefore the problem is reduced to

$$y'' - 2y' + y = 1, \quad y(0) = 1, \; y'(0) = 0.$$

We look for particular solution by the method of undetermine coefficients

$$y_p(t) = q_0$$

Obviously $y_p'' = y_p' = 0$. Hence

$$q_0 = 1$$

and $y_p(t) = 1$. Note that this particular solution satisfies the initial conditions. Hence $y(t) = 1$.