LECTURE 17 (267)

Forced Oscillations and Resonance.

We consider the mass-spring-dashpot system in the situation when we have the external force $F_E = F_0 cos(\omega t)$. **Undamped Forced Oscillations.** First we consider the case when damping constant $c = 0$. Then the motion of our system described by the equation:

$$mx'' + kx = F cos\omega t \tag{1}$$

We remind that the natural frequency is given by formula $\omega_0 = \sqrt{\frac{k}{m}}$ Assume that $\omega \neq \omega_0$. In that case the general solution to equation (1) is

$$x(t) = c_1 cos\omega_0 t + c_2 sin\omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} cos(\omega t).$$

or

$$x(t) = C cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2} cos(\omega t). \tag{2}$$

Resonance. We consider the case when

$$\omega = \omega_0.$$

In that case the equation (1) has the form

$$x'' + \omega_0^2 x = \frac{F_0}{m} cos(\omega_0 t)$$

And the general solution is

$$x(t) = C cos(\omega_0 t - \alpha) + \frac{F_0}{2m\omega_0} t sin(\omega_0 t). \tag{3}$$

Damped Forced Oscillations.

$$mx'' + cx' + kx = F_0 cos(\omega t) \tag{4}$$

We represent the solution to problem (4) in the form

$$x(t) = x_c(t) + x_p(t)$$

Here $x_c(t)$ is a solution to the homogeneous equation. We note that $x_c(t) \to 0$ as $t \to +\infty$. So $x_c(t)$ called the **transient solution.**

We introduce two constants A and B by formulas

$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}$$

Then the particular solution has the form

$$x(t) = Acos(\omega t) + Bsin(\omega t)$$

or

$$x_p(t) = Ccos(\omega t - \alpha), \tag{5}$$

where

$$C = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

The constant $C = C(\omega)$ called forced amplitude. The point where function $C(\omega)$ attain the maximum value is called the particular resonance. If $c \geq \sqrt{2km}$ then $C(\omega)$ is steady decreasing function, but if $c < \sqrt{2km}$ the function C attain a maximum value for some $\hat{\omega}$ less then ω_0.

$$\alpha = \begin{cases} tan^{-1} \frac{c\omega}{k - m\omega^2} & \text{if } k > m\omega^2 \\ \pi + tan^{-1} \frac{c\omega}{k - m\omega^2} & \text{if } k < m\omega^2 \end{cases}$$

Example 1. Solve the initial value problem

$$x'' + 100x = 15cos5t + 20sin5t, x(0) = 25, x'(0) = 0.$$

Solution. We have $m = 1, k = 100, \omega_0 = 10$ So the general solution to O.D.E. is

$$x(t) = c_1 cos(10t) + c_2 sin(10t) + x_p(t).$$

We look for the function $x_p(t)$ as the sum of two functions

$$x_p(t) = x_{p,1}(t) + x_{p,2}(t)$$

Here

$$x''_{p,1} + 100x_{p,1} = 15cos5t, \quad x''_{p,2} + 100x_{p,2} = 20sin5t$$

Using the formula (2) with $F_0 = 15$ and $\omega = 10$ we obtain

$$x_{p,1}(t) = \frac{1}{5}cos5t.$$

Using the method of undetermined coefficients we are looking for the function $x_{p,2}(t)$ in the form

$$x_{p,2}(t) = Acos5t + Bsin5t$$

Then

$$x''_{p,2}(t) = -25Acos5t - 25Bsin5t$$

And

$$x''_{p,2} + 100x_{p,2} = 75(Acos5t + Bsin5t) = 20sin5t.$$

Therefore $A = 0, B = \frac{4}{15}$. The general solution is

$$x(t) = c_1cos(10t) + c_2sin(10t) + \frac{1}{5}cos5t + \frac{4}{15}sin(5t).$$

Now we fix the constants c_1 and c_2.

$$x(0) = 25 = c_1 + \frac{1}{5}$$

Hence $c_1 = \frac{124}{5}$. Then

$$x'(0) = 0 = 10c_2 + \frac{4}{3}$$

Hence $c_2 = -\frac{2}{15}$. The answer is

$$x(t) = \frac{124}{5}cos(10t) - \frac{2}{15}sin(10t) + \frac{1}{5}cos5t + \frac{4}{15}sin(5t).$$

Example 2. Find the particular solution $x_{sp}(t)$ in the form $x_{sp}(t) = Ccos(\omega t - \alpha)$ for the following ordinary differential equation

$$x'' + 3x' + 5x = -4cos5t.$$

Example 3. Derive the steady periodic solution of

$$mx'' + cx' + kx = F_0sin\omega t$$

Solution. Using the method of undetermined coefficients we are looking for a solution in the form

$$x_p(t) = Asin\omega t + Bcos\omega t$$

We need the formulas for the first and the second derivatives of the function $x_p(t)$:

$$x_p'(t) = \omega Acos(\omega t) - \omega Bsin(\omega t), \quad x_p''(t) = -\omega^2(Asin\omega t + Bcos\omega t).$$

Then

$$-m\omega^2(Asin\omega t+Bcos\omega t)+c(\omega Acos(\omega t)-\omega Bsin(\omega t))+k(Asin\omega t+Bcos\omega t) = F_0 sin(\omega t).$$

From this equation we obtain

$$(k - m\omega^2)A - c\omega B = F_0, \quad (k - m\omega^2)B + c\omega A = 0.$$

from the second equation we have

$$B = -\frac{c\omega}{k - m\omega^2}A.$$

Hence

$$(k - m\omega^2)A + \frac{(c\omega)^2}{k - m\omega^2}A = F_0$$

and

$$A = \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + (c\omega)^2}.$$

Then

$$B = -\frac{(c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}.$$