APPLICATIONS OF THE DOUBLE INTEGRALS

Today we consider some applications of the double integrals to mechanics. Suppose we have a lamina which occupies a region D on the plane and made of non homogeneous material. We are looking for the mass of this lamina. First let us introduce the *density function* $\rho(x, y)$. We can do that in the following way. Let a point with coordinates (x, y) be an arbitrary point on the lamina. We consider a square centered at (x, y) with the length of the side h. We denote the mass of this square as Δm and the area as ΔA. (In fact $\Delta A = h^2$.) Then

$$\rho = \lim_{h \to 0} \frac{\Delta m}{\Delta A}.$$

The *moment* of the lamina about the $x - axis$ is given by formula

$$M_x = \int\int_D y\rho(x, y)dA.$$

and the moment about the $y - axis$ is

$$M_y = \int\int_D x\rho(x, y)dA.$$

Definition 1.. *The coordinates $(\overline{x}, \overline{y})$ of the center of mass of a lamina occupying the region D and having density function$\rho(x, y)$ are*

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\int\int_D x\rho(x, y)dA, \quad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\int\int_D y\rho(x, y)dA.$$

The moment of inertia about the $x - axis$ is

$$I_x = \int\int_D y^2\rho(x, y)dA.$$

The moment of inertia about the $y-axis$ is

$$I_y = \int\int_D x^2\rho(x,y)dA.$$

The polar moment of inertia is

$$I_0 = \int\int_D (x^2+y^2)dA.$$

Example 1. Find mass and the center of mass for lamina with the density function $\rho(x,y)=y$. The lamina occupies the region D which is bounded by parabola $y=1-x^2$ and x-axis.

Solution. We can describe the region D as follows

$$D=\{(x,y)|-1\leq x\leq 1, 0\leq y\leq 1-x^2\}.$$

First let us find the mass of lamina

$$m=\int\int_D ydA=\int_{-1}^{1}\int_0^{1-x^2} ydydx=\int_{-1}^{1}\frac{y^2}{2}|_0^{1-x^2}dx=$$
$$\frac{1}{2}\int_{-1}^{1}(1-x^2)^2dx=\frac{1}{2}\int_{-1}^{1}(1-2x^2+x^4)dx=\frac{1}{2}(x-\frac{2x^3}{3}+\frac{x^5}{5})|_{-1}^{1}=$$
$$1-\frac{2}{3}+\frac{1}{5}=\frac{1}{3}+\frac{1}{5}=\frac{8}{15}.$$

Next we find moment of the lamina about the $x-axis$

$$M_x=\int\int_D y\rho(x,y)dA=\int\int_D y^2dA=\int_{-1}^{1}\int_0^{1-x^2} y^2dydx=\int_{-1}^{1}\frac{y^3}{3}|_0^{1-x^2}dx$$
$$=\frac{1}{3}\int_{-1}^{1}(1-x^2)^3dx=$$
$$\frac{1}{3}\int_{-1}^{1}(1+3x^4-3x^2-x^6)dx=\frac{1}{3}(x+\frac{3}{5}x^5-x^3-\frac{x^7}{7})|_{-1}^{1}=\frac{32}{105}.$$

Finally we find moment of the lamina about the $y-axis$

$$M_y=\int\int_D x\rho(x,y)dA=\int\int_D y^2dA=\int_{-1}^{1}\int_0^{1-x^2} xydydx=\int_{-1}^{1}x\frac{y^2}{2}|_0^{1-x^2}dx$$
$$=\frac{1}{2}\int_{-1}^{1}x(1-x^2)^2dx=\frac{1}{12}(x^2-1)^3|_{-1}^{1}=0.$$

Finally

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\int\int_D x\rho(x,y)dA = 0.$$

$$\overline{y} = \frac{M_x}{m} = \frac{1}{m}\int\int_D y\rho(x,y)dA = \frac{32}{105}\frac{15}{8} = \frac{4}{7}.$$

Example 2. Find mass and the center of mass for lamina with the density function $\rho(x,y) = x$. The lamina occupies the region D which is triangle with the vertices $(0,0),(2,0),(1,1)$.

Solution. We can describe the region D as follows

$$D = \{(x,y)|x \le y \le 1, y \le x \le 2-y\}$$

First let us find the mass of lamina

$$m = \int\int_D xdA = \int_0^1\int_y^{2-y} xdxdy = \int_0^1 \frac{x^2}{2}|_y^{2-y}dy =$$

$$\frac{1}{2}\int_0^1((2-y)^2 - y^2)dx = \int_0^1(2-2y)dy = (2y-y^2)|_0^1 = 1.$$

Next we find moment of the lamina about the $x - axis$

$$M_x = \int\int_D y\rho(x,y)dA = \int_0^1\int_y^{2-y} yxdxdy = \int_0^1 y\frac{x^2}{2}|_y^{2-y}dy =$$

$$\int_0^1(2y-2y^2)dy = (y^2 - \frac{2}{3}y^3)|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}.$$

Finally we find moment of the lamina about the $y - axis$

$$M_y = \int\int_D x\rho(x,y)dA = \int\int_D x^2dA = \int_0^1\int_y^{2-y} x^2dxdy = \int_0^1 \frac{x^3}{3}|_y^{2-y}dy =$$

$$\frac{1}{3}\int_0^1((2-y)^3 - y^3)dy = \frac{1}{3}(8-6y^2+12y-2y^3)dy = \frac{1}{3}(8y-2y^3+6y-\frac{y^4}{2})|_0^1 =$$

$$\frac{1}{3}(8-2+6-\frac{1}{2}) = 4 - \frac{1}{6} = \frac{23}{6}.$$

Finally

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\int\int_D x\rho(x,y)dA = \frac{23}{6},$$

$$\overline{y} = \frac{M_x}{m} = \frac{1}{m}\int\int_D y\rho(x,y)dA = \frac{1}{3}.$$