LECTURE 5 (267)

We consider the first order linear ordinary differential equation

$$\frac{dy}{dt} + P(t)y = Q(t) \tag{1'}$$

The general solution to this equation given by formula:

$$y(t) = e^{-\int P(t)dt} \left[\int Q(t) e^{\int P(t)dt} dt + C \right]. \tag{1}$$

Example 1. Solve the initial value problem

$$y' + y = 2, \quad y(0) = 0.$$

Solution. In our case $P(t) = 1, Q(t) = 2$. Then $\int P(t)dt = t$ and $e^{-\int P(t)dt} = e^{-t}$. So $\int Q(t)e^{\int P(t)dt}dt = \int 2e^t dt = 2e^t$. Using the formula (1) we have

$$y(t) = e^{-t}[2e^t + C] = 2e^t + Ce^{-t}.$$

Now we find the constant C.

$$y(0) = 0 = 2 + C$$

Hence $C = -2$ and

$$y(t) = 2e^t - 2e^{-t}.$$

Example 2. Solve the initial value problem

$$y' - 2y = 3e^{2t}, \quad y(0) = 0$$

Solution. In our case $P(t) = -2, Q(t) = 3e^{2t}$. Then $\int P(t)dt = -2t$ and $e^{-\int P(t)dt} = e^{-2t}$. So $\int Q(t)e^{\int P(t)dt}dt = \int 3dt = 3t$. Using the formula (1) we have

$$y(t) = e^{2t}[3t + C] = 3te^{2t} + Ce^{2t}.$$

Now we find the constant C.

$$y(0) = 0 = 0 + C$$

Hence $C = 0$ and

$$y(t) = 3te^{2t}.$$

Example 3. Solve the initial value problem

$$ty' - 3y = t^3, \quad y(1) = 10 \tag{2}$$

Solution. First we reduce equation (2) to the canonical form (1')

$$y' - \frac{3}{t}y = t^2, \quad y(1) = 10.$$

Then $P(t) = -\frac{3}{t}$ and $\int -\frac{3}{t}dt = -3\ln t$, $Q(t) = t^2$. Hence $\int Q(t)e^{\int P(t)dt}dt = \int t^2 e^{-3\ln t}dt = \int \frac{1}{t}dt = \ln t$. Using the formula (1)

$$y(t) = t^3(\ln t + C)$$

Using the initial condition we find a constant C.

$$y(1) = 10 = C.$$

Hence

$$y(t) = t^3(\ln t + 10).$$

Mixture Problems.

We consider a tank containing a salt dissolved in a water. There is both inflow and outflow from this tank. We want to compute the amount of salt in the tank -$x(t)$ at moment t. suppose that a water with concentration c_1 grams of salt per liter flows into the tank at the rate r_1 liters per second, kept thoroughly mixed by stirring and flows out at the constant rate r_0 liters per second. Denote by $V(t)$ the volume of the water in the tank at moment t. Then the concentration of the salt in the water which flows out of the tank is

$$c_0 = \frac{x(t)}{V(t)}.$$

The function $x(t)$ satisfies the O.D.E.

$$\frac{dx}{dt} = r_1c_1 - r_0c_0.$$

or

$$\frac{dx}{dt} = r_1 c_1 - r_0 \frac{x(t)}{V(t)}. \tag{3}$$

Example 4. A tank contains 1000 liters of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of $5L/s$ and the mixture -kept uniform by stirring - is pumped out at the same rate. How long will it be only $10kg$ of salt remains in the tank?

Solution Since the water lows in and out at the same rate it volume in the tank is constant $V(t) = 1000.L$ Note that $c_1 = 0$ so equation (3) has the form

$$\frac{dx}{dt} = -5\frac{x(t)}{1000}$$

The general solution to this equation is

$$x(t) = Ce^{-\frac{t}{200}}.$$

Since $x(0) = 100$ we have

$$x(t) = 100e^{-\frac{t}{200}}.$$

Denote by t_0 the time when only 10 kg of salt left in the tank. Then

$$x(t_0) = 10 = 100e^{-\frac{t_0}{200}}$$

From this equation we have

$$t_0 = 200 \ln 10.$$