LECTURE 32

Review of the Matrix Theory.

Example 1. Find all eigenvalues and eigenvectors for the following matrix:

$$A = \begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix}$$

Solution. We have

$$A - \lambda E = \begin{pmatrix} -2-\lambda & 1 \\ -1 & -4-\lambda \end{pmatrix}$$

Then

$$det(A-\lambda E) = det \begin{pmatrix} -2-\lambda & 1 \\ -1 & -4-\lambda \end{pmatrix} = (-2-\lambda)(-4-\lambda)+1 = \lambda^2+6\lambda+9 = (\lambda+3)^2$$

therefore the matrix A has only one eigenvalue $\lambda = -3$. Lets find eigenvectors of the matrix A We have

$$A + 3E = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Therefore the system $(A+3E)\vec{e} = 0$ Is equivalent to the equation $e_1 + e_2 = 0$. Then vector $\vec{e} = (1, -1)$ is the eigenvector of the matrix A.

Example 2. Find all eigenvalues and eigenvectors for the following matrix:

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

Solution. We have

$$A - \lambda E = \begin{pmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix}$$

Then

$$det(A{-}\lambda E) = det\begin{pmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = (3{-}\lambda)(1{-}\lambda){+}1 = \lambda^2{-}4\lambda{+}4 = (\lambda{-}2)^2.$$

Therefore the matrix A has only one eigenvalue $\lambda = 2$. Lets find eigenvectors of the matrix A We have

$$A - 2E = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

Therefore the system $(A-2E)\vec{e} = 0$ Is equivalent to the equation $e_1 - e_2 = 0$. Then vector $\vec{e} = (1, 1)$ is the eigenvector of the matrix A.

Example 3. Find all eigenvalues and eigenvectors for the following matrix:

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$$

Solution. We have

$$A - \lambda E = \begin{pmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{pmatrix}$$

Then

$$det(A{-}\lambda E) = det\begin{pmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{pmatrix} = (1{-}\lambda)(5{-}\lambda){+}4 = \lambda^2{-}6\lambda{+}9 = (\lambda{-}3)^2.$$

Therefore the matrix A has only one eigenvalue $\lambda = 3$. Lets find eigenvectors of the matrix A We have

$$A - 3E = \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}$$

Therefore the system $(A-3E)\vec{e} = 0$ Is equivalent to the equation $e_1 + e_2 = 0$. Then vector $\vec{e} = (1, -1)$ is the eigenvector of the matrix A.

Example 4. Find all eigenvalues and eigenvectors for the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ -2 & -4 & -1 \end{pmatrix}$$

Solution. We have

$$A - \lambda E = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix}$$

Then

$$det(A-\lambda E) = det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix} = -(\lambda-1)^3.$$

Therefore the matrix A has only one eigenvalue $\lambda = 1$. Lets find eigenvectors of the matrix A We have

$$A - E = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix}$$

Therefore the system $(A-E)\vec{e} = 0$ Is equivalent to the equation $e_1+2e_2+e_3 = 0$. Then we have two linearly independent eigenvectors $\vec{e_1} = (1, 0, -1)$ and $\vec{e_2} = (2, -1, 0)$.

Example 5. Find all eigenvalues and eigenvectors for the following matrix:

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{pmatrix}$$

Solution. We have

$$A - \lambda E = \begin{pmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -4 \\ 0 & 1 & -3-\lambda \end{pmatrix}$$

Then

$$det(A-\lambda E) = det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix} = -(\lambda+1)^3.$$

Therefore the matrix A has only one eigenvalue $\lambda = -1$. Lets find eigenvectors of the matrix A We have

$$A + E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & -1 & -2 \end{pmatrix}$$

Therefore the system $(A+E)\vec{e} = 0$ Is equivalent to the system $e_3 = 0, \quad 2e_2 - 4e_3 = 0, \quad -e_2 - 2e_3 = 0$. Then we have $e_2 = e_3 = 0$ and the matrix A has only one eigenvector $\vec{e} = (1, 0, 0)$.