POWER SERIES

By the **Power Series** Centered at the point a we mean the infinite series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

Here x is the independent variable. We have

Theorem 1. *For given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities*
(i) The series converges only when $x = a$.
(ii) The series converges for all x
(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R is called the **Radius of convergence.** If the power series is convergent for all x we say that the radius of convergence R equal to ∞. If the power series converges only for $x = a$ we say that the radius of convergence equal to zero.

We note that on the interval $(-R+a, a+R)$ our infinite series is absolutely convergent.

Example 1. Find the radius of convergence for the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
Solution. We use the ratio test

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{x}{n+1}| = 0 < 1$$

Therefore the infinite series converges for all x.

Example 2. Find the radius of convergence for the infinite series $\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{n+3}$.
Solution. We use the ratio test

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{2(x-3)(n+3)}{n+4}| = 2|x-3|$$

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By the ratio test we have that the infinite series convergent for

$$|x-3| < \frac{1}{2}.$$

So the radius of convergence is $R = \frac{1}{2}$. The interval of convergence is $(\frac{5}{2}, \frac{7}{2})$.

Example 3. Find the radius of convergence for the infinite series $\sum_{n=0}^{\infty} \frac{(x+1)^n}{n(n+1)}$.

Solution. We use the ratio test

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(x+1)n(n+1)}{(n+2)(n+1)}| = |x+1|$$

By the ratio test we have that the infinite series convergent for

$$|x+1| < 1.$$

So the radius of convergence is $R = 1$. The interval of convergence is $(-2, 0)$.

Example 4. Find the radius of convergence for the infinite series $\sum_{n=0}^{\infty}(2x)^n$.

Solution. We use the ratio test

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(2x)^{n+1}}{(2x)^n}| = \lim_{n\to\infty} |2x| = |2x|$$

By the ratio test we have that the infinite series convergent for

$$|2x| < 1$$

or

$$|x| < \frac{1}{2}.$$

So the radius of convergence $R = \frac{1}{2}$ and the interval of convergence $(-\frac{1}{2}, \frac{1}{2})$.

Example 5. Find the radius of convergence for the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{3^n n^3}$.

Solution. We use the ratio test

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{xn^3}{3(n+1)^3}| = \frac{x}{3}|$$

By the ratio test we have that the infinite series convergent for

$$|x| < 3$$

So the radius of convergence $R = \frac{1}{3}$ and the interval of convergence $(-\frac{1}{3}, \frac{1}{3})$.