# DIVERGENCE THEOREM

**Theorem.** *Let $E$ be a simple solid region whose boundary surface $S$ has positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains $E$. Then*

$$\int\int_S \mathbf{F}\cdot dS = \int\int\int_E div\mathbf{F}dV.$$

**Example 1.** Use the divergence theorem to calculate the surface integral $\int\int_S \mathbf{F}\cdot dS$ where $\mathbf{F} = 3y^2z^3\vec{i} + 9x^2yz^2\vec{j} - 4xy^2\vec{k}$ and the surface $S$ is the cube with vertices $(\pm 1, \pm 1, \pm 1)$.

*Solution.* According to the Divergence Theorem

$$\int\int_S \mathbf{F}\cdot dS = \int\int\int_E div\mathbf{F}dV.$$

Note that

$$div\mathbf{F} = 9x^2z^2$$

and

$$E = \{(x,y,z) | -1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1\}.$$

Therefore

$$\int\int_S \mathbf{F}\cdot dS = \int\int\int_E div\mathbf{F}dV = \int_{-1}^1\int_{-1}^1\int_{-1}^1 9x^2z^2 dzdydx = \int_{-1}^1\int_{-1}^1 3x^2z^3|_{-1}^1 dydx$$

$$= 2\int_{-1}^1\int_{-1}^1 3x^2dydx = 4\int_{-1}^1 3x^2dx = 4x^3|_{-1}^1 = 8.$$

**Example 2.** Use the divergence theorem to calculate the surface integral $\int\int_S \mathbf{F}\cdot dS$ where $\mathbf{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and the surface $S$ is the sphere $x^2 + y^2 + z^2 = 1$.

*Solution.* According to the Divergence Theorem

$$\int\int_S \mathbf{F}\cdot dS = \int\int\int_E div\mathbf{F}dV.$$

Note that

$$div\mathbf{F} = 3(x^2+y^2+z^2)$$

and

$$E = \{(\rho,\theta,\phi)|0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}.$$

Therefore

$$\int\int_S \mathbf{F}\cdot dS = \int\int\int_E div\mathbf{F}dV = 3\int\int\int_E (x^2+y^2+z^2)dV = 3\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^4 sin\phi d\rho d\phi d\theta =$$
$$\frac{3}{5}\int_0^{2\pi}\int_0^{\pi} \rho^5|_0^1 sin\phi d\phi d\theta = \frac{3}{5}\int_0^{2\pi}\int_0^{\pi} sin\phi d\phi d\theta = \frac{3}{5}\int_0^{2\pi} -cos\phi|_0^{\phi} d\theta =$$
$$\frac{6}{5}\int_0^{2\pi} 1 d\theta = \frac{12\pi}{5}.$$

**Example 3.** Use the divergence theorem to calculate the surface integral $\int\int_S \mathbf{F}\cdot dS$ where $\mathbf{F} = (x^3+yz)\vec{i} + x^2y\vec{j} + xy^2\vec{k}$ and $S$ is the surface of the solid bounded by spheres $x^2+y^2+z^2 = 4$ and $x^2+y^2+z^2 = 9$

*Solution.* According to the Divergence Theorem

$$\int\int_S \mathbf{F}\cdot dS = \int\int\int_E div\mathbf{F}dV.$$

Note that

$$div\mathbf{F} = 4x^2$$

and

$$E = \{(\rho,\theta,\phi)|2 \le \rho \le 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}.$$

Therefore

$$\int\int_S \mathbf{F}\cdot dS = \int\int\int_E div\mathbf{F}dV = \int_0^{2\pi}\int_0^{\pi}\int_2^3 4\rho^2 cos^2\theta sin^2\phi \rho^2 sin^2\phi d\rho d\phi d\theta =$$
$$4\int_0^{2\pi}\int_0^{\pi}\int_2^3 \rho^4 cos^2\theta sin^3\phi d\rho d\phi d\theta = \frac{4}{5}\int_0^{2\pi}\int_0^{\pi} \rho^5 cos^2\theta sin^3\phi|_2^3 d\phi d\theta =$$
$$\frac{4}{5}(3^4-2^4)\int_0^{2\pi}\int_0^{\pi} cos^2\theta sin^3\phi d\phi d\theta.$$

Note that the antiderivative for $sin^3\phi$ is

$$\int sin^3\phi d\phi = -\frac{1}{3}(2 + sin^2\phi)cos\phi.$$

Hence

$$\int\int_S \mathbf{F}\cdot dS = \frac{4}{5}(3^4-2^4)\int_0^{2\pi} -\frac{1}{3}(2+sin^2\phi)cos\phi cos^2\theta|_0^\pi d\theta = \frac{16}{15}(3^4-2^4)\int_0^{2\pi} cos^2\theta d\theta = \frac{16}{15}(3^4 - 2^4)\pi.$$

**Example 4.** Use the divergence theorem to calculate the surface integral $\int\int_S \mathbf{F}\cdot dS$ where $\mathbf{F} = ye^{z^2}\vec{i} + y^2\vec{j} + e^{xy}\vec{k}$ and $S$ is the surface of the solid bounded by spheres $x^2 + y^2 = 9$ and $z = 0$ and $z = y - 3$.

*Solution.* According to the Divergence Theorem

$$\int\int_S \mathbf{F}\cdot dS = \int\int\int_E div\mathbf{F}dV.$$

Note that

$$div\mathbf{F} = 2y$$

and

$$E = \{(x, y, z)|(x, y) \in D, 0 \le z \le y - 3\}.$$

Therefore

$$\int\int\int_E 2ydV = \int\int_D\int_0^{y-3} 2ydzdA = \int\int_D yz|_0^{y-3}dA = 2\int\int_D (y^2-3y)dA =$$
$$2\int_0^{2\pi}\int_0^3 (r^2sin^2\theta - 3rsin\theta)rdrd\theta = 2\int_0^{2\pi}(\frac{r^4}{4}sin^2\theta - rsin\theta)|_0^3 d\theta =$$
$$2\int_0^{2\pi}(\frac{3^4}{4}sin^2\theta - 3sin\theta)d\theta = 2(\frac{3^4}{4}(\frac{\theta}{2} - \frac{1}{4}sin(2\theta)) + 3cos\theta)|_0^{2\pi} = \frac{3^4\pi}{2}.$$