LECTURE 18 (267)

Electrical Circuits.

We consider an electric circuit which include:

1. A resistor with a resistance of R ohms
2. An inductor with an inductance of L henries
3. A capacitor with capacitance of C farads.
4. Switch.

(For details please see Figure 2.7.1 on page 170.)

The circuit includes the battery which supplies a voltage $E(t)$ volts at time t. If the switch is closed this results in a current $I(t)$ amperes in the circuit and a charge of $Q(t)$ coulombs on the capacitor at time t.

We assume that the voltage drop for the **inductor** is $L\frac{dI}{dt}$, voltage drop for the **resistor** is RI and the voltage drop for the **capacitor** $\frac{1}{C}Q$.

Kirchoff's law. *The algebraic sum of the voltage drops across the elements in a simple loop of an electrical circuite is equal to applied voltage.*

As the consequence of this law we have

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t) \tag{1}$$

Taking into account that

$$\frac{dQ}{dt} = I(t)$$

we have

$$LQ'' + RQ' + \frac{1}{C}Q = E(t) \tag{2}$$

If we differentiate the equation (1) we obtain

$$LI'' + RI' + \frac{1}{C}I = E'(t) \tag{3}$$

Suppose that

$$E(t) = E_0 sin\omega t$$

Then

$$LI'' + RI' + \frac{1}{C}I = \omega E_0 cos\omega t. \tag{4}$$

The general solution is

$$I = I_{tr} + I_{sp}$$

Here I_{tr} is the transient current, for which we have

$$I_{tr}(t) \to +0 \quad \text{as } t \to +0.$$

and

$$I_{sp}(t) = \frac{E_0 cos(\omega t - \alpha)}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \tag{5}$$

with

$$\alpha = \tan^{-1}\frac{\omega RC}{1 - LC\omega^2}, \quad 0 \le \alpha \le \pi.$$

The quantity in the denominator in (5)

$$Z = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

is called **impendence**. The steady periodic current

$$I_{sp}(t) = \frac{E_0}{Z} cos(\omega t - \alpha)$$

has amplitude

$$I_0 = \frac{E_0}{Z}.$$

We consider the amplitude I_0 as a function of ω: $I_0 = I_0(\omega)$. this function attain the maximum value at the point

$$\omega_m = \frac{1}{\sqrt{LC}}.$$

Example 1. In the circuit of Fig. 2.7.7 with the switch in position 1 suppose that $L = 2, R = 40, E(t) = 100e^{-10t}$ and $I(0) = 0$ Find maximum current in the circuit for $t \ge 0$.

Solution. Since the capacitor in the circuit is absent the equation (2) has a form

$$L\frac{dI}{dt} + RI = E(t).$$

For our particular case we have

$$I' + 20I = 50e^{-t}, I(0) = 0 \tag{6}$$

First we find the general solution to equation (6). The characteristic equation is

$$r + 20 = 0$$

It has one root $r_1 = -20$. So the linearly independent solution to the homogeneous equation is

$$I_{tr}(t) = e^{-20t}$$

Using the method of undetermined coefficients we are looking for the particular solution in the form

$$I_p(t) = q_0 e^{-10t}$$

We observe

$$I_p'(t) = -10q_0 e^{-10t}$$

So

$$I_p' + 20I_p = -10q_0 e^{-10t} + 20q_0 e^{-10t} = 50e^{-t}, I(0) = 0.$$

Hence $q_0 = 5$ and the general solution is

$$I(t) = Ce^{-20t} + 5e^{-10t}$$

Using the initial condition we find the constant C:

$$I(0) = 0 = C + 5, \quad C = -5.$$

Hence

$$I(t) = 5(e^{-10t} - e^{-20t}) = 5e^{-10t}(1 - e^{-10t}).$$

The function $I(t)$ is positive for all $t > 0$. We observe that

$$\lim_{t \to +\infty} I(t) = 0.$$

So there exists at least one point where the function $I(t)$ attain it maximum. By Fermat theorem at this point $I'(t) = 0$. Lets solve equation $I'(t) = 0$. We have

$$I'(t) = 5(-10e^{-10t} + 20e^{-20t}) = 0.$$

This equation has only one solution $t = \ln(2)/10$. So the maximal value of current in $\frac{5}{4}$.

Example 2. In the circuit of Fig 2.7.7 suppose that $L = 5H$ $R = 25\Omega$ and that the source of E is a battery supplying $100V$ to the circuit. Suppose that the switch has been in position 1 for long time so that steady current of $4A$ is following in the circuit. At time $t = 0$ the switch is thrown to position 2, so that $I(0) = 4$ and $E = 0$ for $t \geq 0$. Find $I(t)$.

Example 3. We have an electric circuit which consists of resistor $R = 16\Omega$, inductor $L = 2H$ capacitor $C = 0.02F$ The battery generate $E(t) = 100V$ Initial current $I(0) = 0$ and the initial charge is $Q(0) = 5$.

Solution. Equation (3) has the form

$$2I'' + 16I' + 50I = 0$$

The roots of the characteristic equation are

$$r_{1,2} = -4 \pm 3i$$

Hence the general solution is

$$I(t) = C_1 e^{-4t} cos(3t) + C_2 e^{-4t} sin(3t). \tag{7}$$

Using the initial condition we obtain

$$I(0) = 0 = C_1$$

So

$$I(t) = C_2 e^{-4t} sin(3t).$$

From equation (1) we have

$$I'(0) = \frac{1}{L}(E(0) - RI(0) - \frac{1}{C}Q(0)) = -75$$

From (7) we find

$$I'(t) = -4tC_2 e^{-4t} sin(3t) + 3C_2 e^{-4t} cos(3t)$$

Hence

$$-75 = 3C_3, \quad C_3 = -25$$

and

$$I(t) = -25e^{-4t} sin(3t).$$