LECTURE 11 (267)

Mechanical Vibrations.

We consider the following model. To the spring suspended vertically form a fixed support. We put the center of coordinate system x at the lower end of the spring. Then we attach the mass m to the lower end of the spring. According to the Hooke's low the spring has a restorative force F_s given by formula

$$F_s = -kx$$

The positive constant k is called the spring constant. On the other hand the gravitational force F_G is acting on the spring. We remind that near the earth's surface the gravitational force given by formula

$$F_G = -mg.$$

Also we assume that there is a resistance force

$$F_R = -c\frac{dx}{dt}.$$

the positive constant c called the damping constant. So according to the second Newton's law the equation of the motion is

$$mx'' = F_G + F_s + F_R. \tag{1}$$

The mass m would stretch the spring at distance s_0. We can find s_0 from the equation

$$F_G + F_s = 0$$

This equation imply

$$s_0 = mg/k.$$

Next we introduce the new coordinate system y and put the center of this system at the lower end of the stretched spring. Then equation (1) can be rewritten as

$$my'' + cy' + ky = 0 \tag{2}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

We analyze the solutions to equation (2)

Free undamped Motion. In this case the damping constant c is zero and equation (2) has the form

$$my'' + ky = 0$$

Introducing the constant ω_0 by the formula $\omega_0 = \sqrt{\frac{k}{m}}$ we rewrite this equation as

$$y'' + \omega_0^2 y = 0 \tag{3}$$

The general solution to equation (3) is

$$y(t) = Acos(\omega_0 t) + Bsin(\omega_0 t). \tag{4}$$

We try to simplify the formula (3) We set $C = \sqrt{A^2 + B^2}$,and introduce the angle α as a solution to the system of equations $cos\alpha = \frac{A}{C}, sin\alpha = \frac{B}{C}$.For the angle α we have the explicit formula

$$\alpha = \begin{cases} tan^{-1}(B/A) & \text{if } A > 0, B > 0 \\ \pi + tan^{-1}(B/A) & \text{if } A < 0 \\ 2\pi + tan^{-1}(B/A) & \text{if } A > 0, B < 0 \end{cases}$$

We can transform formula (4) to the form

$$y(t) = cos(\omega_0 t - \alpha) \tag{5}$$

C-Amplitude, $\omega_0 = \sqrt{\frac{k}{m}}$ -circular frequency, α-phase angle, $\nu = \frac{1}{T} = \frac{\omega_0}{2\pi}$ -frequency, $T = \frac{2\pi}{\omega_0}$ period.

Free Damped Motion.

Now we consider the situation when the damping constant is positive. Introducing the constant$p$ by formula $p = \frac{c}{2m}$ we rewrite the equation (2) in the form

$$y'' + 2py' + \omega_0^2 y = 0. \tag{6}$$

the characteristic equation which corresponds to ODE (5) has two roots

$$r_{1,2} = -p \pm (p^2 - \omega_0^2)^{\frac{1}{2}}$$

These roots might be real, complex or real double. The situation depends on the *sign* of $p^2 - \omega_0^2$. We remind

$$p^2 - \omega_0^2 = \frac{c^2 - 4km}{4m^2}$$

Critical damping $c_{cr} = \sqrt{4km}$.

Over damped case. $c > c_{cr}$ The general solution to equation (6) is

$$y(t) = c_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Critically damped case $c = c_{cr}$. The general solution to equation (6) is

$$y(t) = e^{-pt}(c_1 + c_2 t).$$

Under damped case $c < c_{cr}$. The general solution to equation (6) is

$$y(t) = e^{-pt}(Acos(\omega_1 t) + Bsin(\omega_1 t)) \tag{7}$$

We transform formula (7) to the form similar to (5)

$$y(t) = Ce^{-pt}cos(\omega_1 t - \alpha)$$

where

$$C = \sqrt{A^2 + B^2}, \quad cos(\alpha) = \frac{A}{C}, \quad sin\alpha = \frac{B}{C}$$

and $\omega_1 = \sqrt{\omega_0^2 - p^2}$. Here ω_1 is the circular frequency, $T_1 = \frac{2\pi}{\omega_1}$ pseudo period, Ce^{-pt}- time-varying amplitude.

Example 1.A mass of $3kg$ attached to the end of the spring that is stretched $0.2m$ by a force of $15N$. It is set in motion with initial position $x_0 = 0$ and velocity $v_0 = -10m/s$. Find the amplitude period and frequency of the resulting motion.

Solution. First we find the spring constant k. We have

$$k = 15/0.2 = 75.$$

The circular frequency

$$\omega_0 = \sqrt{\frac{k}{m}} = 5.$$

The period is

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{5}.$$

Frequency

$$\nu = \frac{1}{T} = \frac{5}{2\pi}.$$

Next we find the amplitude C. The general solution is

$$y(t) = Acos(5t) + Bsin(5t).$$

Then $y(0) = 0 = A$ and $y'(0) = -10 = 5B$. Hence $B = -2$ and $C = 2$.