EXAM 4

Problem 1. Solve the initial value problem

$$X' = AX, \quad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Solution. First we find an eigenvalues of the matrix A We have $det(A - \lambda E) = (2-\lambda)^2$. So we have one eigenvalue $\lambda = 2$ of multiplicity $k = 2$. Lets find eigenvectors associated with this eigenvalue We have

$$A - 2E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

So the system $(A - 2E)\vec{e} = 0$ is equivalent to equation $e_2 = 0$. And we have only one eigenvector $v_1 = (1, 0)$. Lets find vector v_2. We have $(A - 2E)v_2 = v_1$. So $v_2 = (0, 1)$ the general solution is

$$X(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

Now lets find the constants C_1, C_2.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore $C_1 = 0, C_2 = 1$. The answer is

$$X(t) = e^{2t} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

Problem 2. Find a general solution for the system of linear ordinary differential equations

$$X' = AX, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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Solution. First we find an eigenvalues of the matrix A We have $det(A - \lambda E) = \lambda^2 + 1$. So we have two complex eigenvalues $\lambda_{1,2} = \pm i$ of multiplicity $k = 1$. Lets find eigenvectors associated with the eigenvalue $\lambda_1 = i$ We have

$$A - iE = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}.$$

The system $(A - iE)e = 0$ equivalent to equation $-ie_1 + e_2 = 0$. So $\vec{e} = (1, i) = (1, 0) + i(0, 1)$. The general solition is

$$X(t) = C_1 \begin{pmatrix} cos(t) \\ -sin(t) \end{pmatrix} + C_2 \begin{pmatrix} sin(t) \\ cos(t) \end{pmatrix}$$

Problem 3. Find a general solution for the system of linear ordinary differential equations

$$X' = AX, \quad A = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Solution. First we find an eigenvalues of the matrix A We have $det(A - \lambda E) = -\lambda(\lambda - 1)^2$. So we have two eigenvalues $\lambda_1 = 0$ of multiplicity $k = 1$ and $\lambda_2 = 1$ of multiplicity $k = 2$. Lets find eigenvectors associated with the eigenvalue $\lambda_1 = 0$. We have

$$\begin{cases} 2e_1 + e_2 - e_3 = 0 \\ -e_1 + e_3 = 0 \\ e_1 + e_2 = 0 \end{cases}$$

We subtract from the first equation the third equaton multiplied by -2 and we subtract from the second equation the third one.

$$\begin{cases} -e_2 - e_3 = 0 \\ e_2 + e_3 = 0 \\ e_1 + e_2 = 0 \end{cases}$$

The first equation is the second one multiplied by -1.

$$\begin{cases} e_2 + e_3 = 0 \\ e_1 + e_2 = 0 \end{cases}$$

$e = (1, -1, 1)$.

$$A - E = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

So the system $(A - E)\vec{e} = 0$ consists of one equation $e_1 + e_2 - e_3 = 0$. It has two linearly independent solutions $(1, 0, 1)$ $(0, 1, 1)$. The general solution is

$$X(t) = C_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 4. For the matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

find $A + 2B, 3A^2 - B$.

Solution

$$A + 2B = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}.$$

Next

$$A^2 = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Therefore

$$3A^2 = \begin{pmatrix} 6 & 3 & -3 \\ -3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}.$$

Hence

$$3A^2 - B = \begin{pmatrix} 6 & 2 & -3 \\ -4 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}.$$