EXAM 3 (267)

Problem 1. Solve the initial value problem

$$y'' + 2y' + 10y = 2\delta(t-1) \quad y(0)=0,\ y'(0)=0.$$

Solution. Taking the Laplace transform of the ordinary differential equation we obtain

$$s^2F(s) + 2sF(s) + 10F = 2e^{-s}$$

Hence

$$F(s) = \frac{2e^{-s}}{(s+1)^2+3^2}$$

Note that

$$\mathcal{L}^{-1}\{\frac{2}{(s+1)^2+3^2}\} = \frac{2}{3}e^{-t}sin(3t).$$

Therefore

$$y(t) = \frac{2}{3}u(t-1)e^{-(t-1)}sin(3(t-1)).$$

Problem 2. Find the Laplace transform of the function

$$g(t) = \begin{cases} 1+2t & \text{for } t \in [0,\sqrt{3}) \\ 3t & \text{for } t \in [\sqrt{3},+\infty) \end{cases}$$

Solution. We represent the function $g(t)$ in the form

$$g(t) = (1+2t)(1-u(t-\sqrt{3})) + 3tu(t-\sqrt{3}).$$

Therefore

$$\mathcal{L}\{g\} = \frac{1}{s} - \frac{e^{-\sqrt{3}s}}{s} + \frac{2}{s^2} + 2\frac{d}{ds}\frac{e^{-\sqrt{3}s}}{s} - 3\frac{d}{ds}\frac{e^{-\sqrt{3}s}}{s}$$

Problem 3. Find the inverse Laplace transform for the function

$$G(s) = \frac{1}{(1+s)(4+s^2)}.$$

Solution. We try represent the function $G(s)$ as the sum of simple functions

$$G(s) = \frac{1}{(1+s)(4+s^2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4} = \frac{4A + As^2 + Bs^2 + Cs + Bs + C}{(1+s)(4+s^2)}.$$

We have the system of linear equations for unknown coefficients A, B, C:

$$\begin{cases} A + B = 0 \\ C + B = 0 \\ 4A + C = 1 \end{cases}$$

Solution to this system is

$$A = \frac{1}{5}, B = -\frac{1}{5}, C = \frac{1}{5}.$$

Therefore

$$G(s) = \frac{1}{5(s+1)} - \frac{1}{5}\frac{s}{s^2+4} + \frac{1}{10}\frac{2}{s^2+4}$$

Now we can easily find the inverse transform of the function $G(s)$:

$$\mathcal{L}^{-1}\{G(s)\} = \frac{e^{-t}}{5} + \frac{1}{10} sin(2t) - \frac{1}{5} cos(2t).$$

Problem 4. We consider the mass- spring-dashpot system with $m = 1, c = 6, k = 9$ and external force $f(t)$,

$$x'' + 6x' + 9x = f(t) \quad x(0) = 0, \quad x'(0) = 1,$$

where

$$f(t) = \begin{cases} 1 & \text{for } t \in [0, 2\pi) \\ 0 & \text{for } t \in [2\pi, +\infty) \end{cases}$$

Find $x(t)$.

Solution. Taking the Laplace transform of the ordinary differential equation we obtain

$$s^2F(s)-sx(0)-x'(0)+6sF(s)-6x(0)+9F(s) = \mathcal{L}\{f\} = \mathcal{L}\{1-u(t-2\pi))\} = \frac{1}{s}-\frac{e^{-2\pi s}}{s}.$$

Using the initial conditions we have

$$(s+3)^2F(s) = 1+\frac{1}{s}-\frac{e^{-2\pi s}}{s}$$

Therefore

$$F(s) = \frac{1}{(s+3)^2}+\frac{1}{s(s+3)^2}-\frac{e^{-2\pi s}}{s(s+3)^2}$$

Note that

$$\mathcal{L}^{-1}\{\frac{1}{(s+3)^2}\} = te^{-3t}.$$

Also

$$\mathcal{L}^{-1}\{\frac{1}{s(s+3)^2}\} = \int_0^t \tau e^{-3\tau}d\tau = -\frac{1}{9}(3\tau+1)e^{-3\tau}|_0^t = -\frac{1}{9}(3t+1)e^{-3t}+\frac{1}{9}.$$

Then we have

$$\mathcal{L}^{-1}\{\frac{e^{-2\pi s}}{s(s+3)^2}\} = u(t-2\pi)\frac{1}{9}(-(3(t-2\pi)+1)e^{-3(t-2\pi)}+1)$$

Finally we have

$$x(t) = te^{-3t}-\frac{1}{9}(3t+1)e^{-3t}+\frac{1}{9}+u(t-2\pi)(\frac{1}{9}(3(t-2\pi)+1)e^{-3(t-2\pi)}-\frac{1}{9})$$