SOLUTIONS FOR EXAM 2

Problem 1.
The characteristic polynomial is $r^2 - 2r + 1 = (r-1)^2$. It has only one root $r_1 = 1$ therefore the general solution to homogeneous equation is

$$y_h(t) = c_1 e^t + c_2 t e^t.$$

Next we try to find a particular solution

$$y_p(t) = q_0 + q_1 t$$

then $y_p' = q_1, y_p'' = 0$ hence

$$y_p'' - 2y_p' + y_p = -2q_1 + q_0 + q_1 t = t + 1$$

Hence $q_1 = 1, -2q_1 + q_0 = 1$ So $q_0 = 3$. The general solution to equation is

$$y(t) = y_h(t) + y_p(t) = c_1 e^t + c_2 t e^t + t + 3.$$

Using the initial conditions we obtain

$$y(t) = t + 3.$$

Problem 2. The characteristic polynomial is $r^2 + r + 1$. then the roots are $r_{1,2} = -\frac{1}{2} \pm \sqrt{3/4} i$. the general solution to the homogeneous equation is

$$y_h(t) = e^{\frac{-t}{2}}(c_1 cos(\sqrt{3/4}t) + c_2 sin(\sqrt{3/4}t)).$$

Now let's find a particular solution. We split a function into the sum of two functions

$$y_p(t) = y_{p,1}(t) + y_{p,2}(t)$$

Here

$$yp,1'' + y_{p,1}' + y_{p,1} = cos(t) \quad yp,2'' + y_{p,2}' + y_{p,2} = 1.$$

We are looking for the function $y_{p,2}$ in the form $y_{p,2} = q_0$ Hence $y_{p,2}' = y_{p,2}'' = 0$ and $q_0 = 1$ So

$$y_{p,2} = 1.$$

Next we are looking for the function $y_{p,1}$ in the form

$$y_{p,1} = q_0 cos(t) + r_0 sin(t)$$

We have

$$y_{p,1}' = -q_0 sin(t) + r_0 cos(t), \quad y_{p,1}'' = -y_{p,1}$$

So

$$-q_0 sin(t) + r_0 cos(t) = cos(t)$$

and we have

$$q_0 = 0, \quad r_0 = 1$$

That gives

$$y_{p,1} = sin(t)$$

The answer is

$$y(t) = e^{\frac{-t}{2}}(c_1 cos(\sqrt{3/4}t) + c_2 sin(\sqrt{3/4}t)) + sin(t) + 1$$

Problem 3. We are using a formula on variation of parameters. The characteristic polynomial is $r^2 + 9$. So $r_{1,2} = \pm 3i$ and we set $y_1(t) = cos(3t), y_2(t) = sin(3t)$. Wronskian

$$W(y_1, y_2) = 3$$

Therefore

$$y_p(t) = -2cos(3t)\int \frac{sin(3t)}{3cos(3t)}dt + 2sin(3t)\int \frac{cos(3t)}{3cos(3t)}dt = -\frac{2}{9}cos(3t)\ln(cos(3t)) + \frac{2}{3}sin(3t).$$

The answer is

$$y(t) = C_1 cos(3t) + C_2 sin(3t) - \frac{2}{9}cos(3t)\ln(cos(3t)) + \frac{2}{3}sin(3t).$$

Problem 4. We have $k/5 = 15$. Then $k = 75$. The equation is

$$3y'' + 75y = 0.$$

Hence $\omega_0 = 5$ $T = 2\pi/\omega_0 = 2\pi/5$. Finally $y(t) = C_1 cos(5t) + C_2 sin(5t)$. Since $y(0) = 0$ we have $C_1 = 0$ Then $C_2 = -2$. Hence $C = 2$.