SOLUTIONS FOR THE SAMPLE EXAM 4.

Problem 1. Note that

$$\ln(n)(1-cos(\frac{1}{n})) = \ln(n)(1-cos^2(\frac{1}{2n})+sin^2(\frac{1}{2n})) = 2\ln(n)sin^2(\frac{1}{2n}) = 2\frac{\ln(n)}{(2n)^2}\frac{sin^2(\frac{1}{2n})}{\left(\frac{1}{2n}\right)^2}.$$

Since

$$\lim_{n\to+\infty}\frac{sin^2(\frac{1}{2n})}{\left(\frac{1}{2n}\right)^2} = 1$$

for all sufficiently large n starting from some N we have

$$|2\frac{\ln(n)}{(2n)^2}\frac{sin^2(\frac{1}{2n})}{\left(\frac{1}{2n}\right)^2}| \leq |\frac{\ln(n)}{n^2}| \leq \frac{1}{n^{\frac{3}{2}}}.$$

The infinite series $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ is the p-series with $p=\frac{3}{2}$. So it is convergent.By comparison test our infinite series is convergent also.

Problem 2. We use the ratio test

$$|\frac{a_{n+1}}{a_n}| = \frac{n+1}{e^{2n+1}}.$$

Therefore

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty}\frac{n+1}{e^{2n+1}} = 0 < 0.$$

Hence our infinite series is convergent by the Ratio Test.

Problem 3. Note that $sin(x)+2 \geq 1$ therefore $\frac{(sin(x)+2)^n}{\sqrt{n}} \geq \frac{1}{\sqrt{n}}$. The infinite series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is the power series with $p=\frac{1}{2}$ so it is divergent. Therefore our infinite series is divergent for all x.

Problem 4. We use the ratio test:

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty}|\frac{\ln(n+1)}{\ln n}||x-1|^2 = |x-1|^2.$$

The series if convergent for all x such that $|x-1|^2 < 1$ and divergent for all x such that $|x-1|^2 > 1$. Hence the radius of convergence $R=1$.