SOLUTIONS FOR THE SAMPLE EXAM 4.

Problem 1. Instead of looking for the limit of the sequence $\{a_n\}$ we try to find the limit of the sequence $\ln(a_n)$.

$$\ln(a_n) = 2n\ln(1+\frac{1}{2n}+\frac{1}{n^2}) = \frac{\ln(1+\frac{1}{2n}+\frac{1}{n^2})}{\frac{1}{2n}}.$$

Therefore

$$\lim_{n\to\infty}\ln(a_n) = \lim_{n\to\infty}\frac{\ln(1+\frac{1}{2n}+\frac{1}{n^2})}{\frac{1}{2n}} = \lim_{x\to 0}\frac{\ln(1+x+4x^2)}{x} = \lim_{x\to 0}\frac{1+8x}{1+x+4x^2} = 1.$$

Hence

$$\lim_{n\to\infty} a_n = e$$

Problem 2. Note that for all sufficiently large positive n we have the inequality

$$\frac{1}{(\ln(n)+1)^2} \geq \frac{1}{\sqrt{n}}.$$

Since the infinite series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is the $p-$series with $p=\frac{1}{2}$ it is divergent. So by comparison test our infinite series is divergent also.

Problem 3. We use the ratio test.

$$a_n = \frac{n^2(x-4)^n}{n!}, \quad a_{n+1} = \frac{(n+1)^2(x-4)^{(n+1)}}{(n+1)!}.$$

Next we find the limit

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty}\frac{(n+1)^2}{n^2}\frac{|x-4|}{(n+1)} = 0.$$

So the radius of convergence is $R=\infty$.

Problem 4.

$$\frac{cos(2t)-1}{t} = \frac{\sum_{n=0}^{\infty}(-1)^n\frac{(2t)^{2n}}{(2n)!}-1}{t} = \sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}t^{2n-1}}{(2n)!}.$$

Therefore

$$\int_0^x\frac{cos(2t)-1}{t}dt = \int_0^x\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}t^{2n-1}}{(2n)!}dt = \sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}x^{2n}}{(2n)!2n}$$