QUIZ 3 (267)

Problem 1. Find the inverse Laplace transform for the function $F(s) = \ln\left(\frac{s-1}{s}\right)$.

Solution. We have

$$F(s) = \ln\left(\frac{s-1}{s}\right) = \int_s^{+\infty} (\frac{1}{\sigma} - \frac{1}{\sigma-1})d\sigma = \int_s^{+\infty} G(\sigma)d\sigma,$$

where $G(\sigma) = \frac{1}{\sigma} - \frac{1}{\sigma-1}$. Note that

$$\mathcal{L}^{-1}\{G(s)\} = 1 - e^t.$$

Therfore

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1-e^t}{t}.$$

Problem 2. Solve the initial value problem:

$$x'' + 4x' + 4x = 2\delta(t-\pi), \quad x(0) = 0,\ x'(0) = 0.$$

Solution. Taking the Laplace transform of the ordinary differential equation we have

$$(s+2)^2 F(s) = 2e^{-s\pi}.$$

Therefore

$$F(s) = \frac{2e^{-s\pi}}{(s+2)^2}.$$

Note that

$$\mathcal{L}^{-1}\{\frac{1}{(s+2)^2}\} = e^{-2t}t.$$

Hence

$$x(t) = 2u(t-\pi)(t-\pi)e^{-2(t-\pi)}.$$

Problem 3. Consider the mass-spring dashpot system with position function $x(t)$ satisfying the equation

$$mx'' + cx' + kx = f(t), \quad x(0) = 0,\ x'(0) = 0,$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where $m = 1, c = 0, k = 4$ and $f(t) = cos(t)u(t-1)$. Find $x(t)$.

Solution. First we note that

$$\mathcal{L}\{f\} = \int_1^{+\infty} e^{-st} cos(t)dt = \frac{e^{-st}}{s^2+1}(-scos(t)+sin(t))|_1^{+\infty} = \frac{e^{-s}}{s^2+1}(-sin(1)+scos(1)).$$

Now taking the Laplace transform of the ordinary differential equation we have

$$(s^2+4)F(s) = \frac{e^{-s}}{s^2+1}(-sin(1)+scos(1)).$$

Or

$$F(s) = \frac{e^{-s}}{(s^2+4)(s^2+1)}(-sin(1)+scos(1)).$$

Next we note that

$$\mathcal{L}^{-1}\{\frac{(-sin(1)+scos(1))}{(s^2+4)(s^2+1)}\} = \frac{1}{2}\int_0^t (-sin(1)sin(\tau)sin(2(t-\tau))+cos(1)cos(\tau)sin(2(t-\tau))d\tau$$

Hence

$$x(t) = u(t-1)\frac{1}{2}\int_0^{t-1} (-sin(1)sin(\tau)sin(2(t-1-\tau))+cos(1)cos(\tau)sin(2(t-1-\tau))d\tau$$

Problem 4. Find the Laplace transform for the function $f(t) = t^2+1+t$ if $0 \leq t < 2$ and $f(t) = 1$ if $t \geq 2$.

Solution. We represent the function f in the form

$$f(t) = 1 + (t^2+t)(1-u(t-2)).$$

Thaking the Laplace transform of this function we have

$$\mathcal{L}\{f(t)\} = \frac{1}{s} + \frac{2}{s^3} + \frac{1}{s^2} - \frac{d^2}{ds^2}(\frac{e^{-2s}}{s}) + \frac{d}{ds}(\frac{e^{-2s}}{s}).$$