EXAM 1 (FALL 2003)

Problem 1. *(25 points)* Find a general solution to the ordinary differential equation

$$(y^3 + \ln x)\frac{dy}{dx} = -\frac{y}{x}. \tag{1}$$

Solution. This ordinary differential equation is exact. Really we rewrite it in the form

$$\frac{y}{x} + (y^3 + \ln x)\frac{dy}{dx} = 0.$$

Then

$$M(x,y) = \frac{y}{x}, \quad N(x,y) = (y^3 + \ln x)$$

Obviously

$$M_y = \frac{1}{x} = N_x = \frac{1}{x}.$$

Then there exist a function $F(x,y)$ such that

$$F_x = \frac{y}{x}, \quad F_y = (y^3 + \ln x)$$

Taking the antiderivatives of these equations respect to x and y we have

$$F(x,y) = y\ln(x) + C_1(y), \quad F(x,y) = \frac{y^4}{4} + y\ln x + C_2(x)$$

Hence

$$y\ln(x) + C_1(y) = \frac{y^4}{4} + y\ln x + C_2(x)$$

So

$$C_1(y) - \frac{y^4}{4} = C_2(x)$$

Since in the left hand side of this equation we have a function which depends only on the variable y and in the right hand side we have a function which depends only on the variable x we have

$$C_1(y) - \frac{y^4}{4} = C_2(x) = C.$$

Hence

$$F(x, y) = \frac{y^4}{4} + y \ln x + C.$$

If $y(x)$ is a solution to the ordinary differential equation (1) then there exist a constant C such that

$$\frac{y^4(x)}{4} + y(x) \ln x = C.$$

Problem 2 *(25 points)* Solve the initial value problem

$$t\frac{dy}{dt} - 2y = 2t^4, \quad y(1) = 1. \tag{2}$$

Solution. We divide the right and the left hand sides of equation (2) by t.

$$\frac{dy}{dt} - \frac{2y}{t} = 2t^3$$

Then

$$P(t) = -\frac{2}{t}, \int P(t)dt = -2\ln t, \quad Q(t) = 2t^3, \int Q(t)e^{\int P(t)dt}dt = \int 2t^3\frac{1}{t^2}dt = \int 2tdt = t^2.$$

The general solution to equation (2) given by formula

$$y(t) = t^4 + Ct^2.$$

Using the initial condition $y(1) = 1$ we have

$$1 = 1 + C$$

So $C = 0$. Finally we have: $y(t) = t^4$.

Problem 3. *(25 points)* A tank initially contains 60 gal of pure water. Brine containing $1lb$ of salt per gallon enters the tank at $2gal/min$ and leaves the tank at $2gal/min$ perfectly mixed. Find a time moment when the concentration of a salt in the brine leaving the tank is $0.5lb$ of salt per gallon.

Solution. First we note that the volume of brine in the tank $V(t)$ is constant since a brine enters the tank at $2gal/min$ and leaves the tank at $2gal/min$. Denote by $x(t)$ the mass of the salt in the tank at moment t. Then we have $c_1 = 1, r_1 = 2, c_0 = x(t)/V(t) = x(t)/60$. Denote by t_0 the time moment when the concentration of the salt in the brine which leave the tank is $0.5lb$. Then

$$x(t_0) = 30lb.$$

Now we set up the equation for $x(t)$:

$$\frac{dx}{dt} = 2 - \frac{x(t)}{30}.$$

The general solution to this equation is

$$x(t) = 60 + Ce^{-\frac{t}{30}}.$$

Since initially we have only a pure water in the tank $C = -60$. Then

$$x(t) = 60 - 60e^{-\frac{t}{30}}.$$

Now, let us try to find t_0:

$$x(t_0) = 30 = 60 - 60e^{-\frac{t_0}{30}}$$

Hence

$$t_0 = 30 \ln 2.$$

Problem 4. *(25 points)* Consider the population of rabbit with the death rate δ equal zero and the birth rate which is proportional to $P(t)$. (Here $P(t)$ is the number of rabbits in the population in the moment t.) Initially $P(0) = 100$. After six months the population of rabbits doubled. When it explode?(In other words, find a time moment t_0 such that $P(t_0) = +\infty$.)

Solution. We set up an equation for $P(t)$:

$$\frac{dP}{dt} = kP^2$$

This equation is separable and the general solution to this equation is

$$-\frac{1}{P(t)} = kt + C.$$

Since $P(0) = 100$ we have $C = -\frac{1}{100}$ and

$$-\frac{1}{P(t)} = kt - \frac{1}{100}.$$

Next

$$P(t) = \frac{100}{1 - 100kt}.$$

and

$$P(6) = 200 = \frac{100}{1 - 600k}.$$

Hence $k = \frac{1}{1200}$ and $t_0 = 12$.