EXAM 2 (FALL 2003)

Problem 1. Find a general solution to the ordinary differential equation

$$y^{(3)} - 4y'' + 4y' - y = 3$$

Solution. We factorize the characteristic polynomial as

$$r^3 - 4r^2 + 4r - 1 = (r-1)(r^2 - 3r + 1) = (r-1)(r - (\frac{3}{2} + \sqrt{\frac{5}{2}}))(r - (\frac{3}{2} - \sqrt{\frac{5}{2}}).$$

so the general solution to the homogeneous equation is

$$Y(t) = C_1 e^t + C_2 e^{(\frac{3}{2} + \sqrt{\frac{5}{2}})t} + C_3 e^{(\frac{3}{2} - \sqrt{\frac{5}{2}})t}.$$

Applying the method of undetermined coefficients we are looking for a particular solution in the form

$$y_p(t) = q_0.$$

Since $y_p^{(3)}(t) = y_p''(t) = y_p'(t) = 0$ we have $q_0 = -3$. So the general solution is

$$y(t) = C_1 e^t + C_2 e^{(\frac{3}{2} + \sqrt{\frac{5}{2}})t} + C_3 e^{(\frac{3}{2} - \sqrt{\frac{5}{2}})t} - 3.$$

Problem 2. Solve the initial value problem

$$y'' + 4y' + 4y = e^{-2t}, \quad y(0) = 0, \; y'(0) = 0.$$

Solution. First we find a general solution to the O.D.E. The characteristic polynomial is $(r + 4r + 4) = (r+2)^2$ So we have only one root $r = -2$ of multiplicity two. The general solution to the homogeneous equation is

$$Y(t) = C_1 e^{-2t} + C_2 t e^{-2t}.$$

Applying the method of undetermined coefficients we are looking for the particular solution in the form

$$y_p(t) = q_0 t^2 e^{-2t}$$

Obviously

$$q_0 = \frac{1}{2}.$$

So the general solution is

$$y(t) = C_1 e^{-2t} + C_2 t e^{-2t} + \frac{t^2 e^{-2t}}{2}.$$

Since $y(0) = 0$ we have $C_1 = 0$. From the second initial condition $y'(0) = 0$ we obtain $C_2 = 0$. So the solution to the initial value problem is

$$y(t) = \frac{t^2 e^{-2t}}{2}.$$

Problem 3. Solve the endpoint problem

$$y'' - y = e^t, \quad y(0) = 0, \; y(1) = 1$$

Solution. First we find the general solution to the O.D.E. The characteristic polynomial is

$$r^2 - 1 = (r-1)(r+1).$$

So the general solution to the homogeneous equation is

$$Y(t) = C_1 e^t + C_2 e^{-t}.$$

Using the method of undetermined coefficients we are looking for a particular solution in the form $y_p(t) = q_0 t e^t$. Obviously $q_0 = \frac{1}{2}$. So the general solution is

$$y(t) = C_1 e^t + C_2 e^{-t} + \frac{te^t}{2}.$$

Since $y(0) = 0$ we have $C_1 = -C_2$. Hence

$$y(t) = C_1(e^t - e^{-t}) + \frac{te^t}{2}.$$

Using the second boundary condition $y(1) = 1$ we have

$$1 = C_1(e - e^{-1}) + \frac{e}{2}.$$

So

$$C_1 = \frac{1 - \frac{e}{2}}{(e - e^{-1})}.$$

The solution to the endpoint problem is

$$y(t) = \frac{1 - \frac{e}{2}}{(e - e^{-1})}(e^t - e^{-t}) + \frac{te^t}{2}.$$

Problem 4. Find a general solution to the ordinary differential equation

$$y'' + 2y' + y = \frac{1}{t^3 e^t} + 3.$$

Solution. The characteristic polynomial is

$$r + 2r + 1 = (r + 1)^2.$$

So we have one root $r = -1$ of multiplicity two. To this root corresponds two solutions $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$. So the general solution to the homogeneous equation is

$$Y(t) = C_1 e^{-t} + C_2 te^{-t}.$$

We split our particular solution into the sum of two functions

$$y_p(t) = y_{p,1}(t) + y_{p,2}(t)$$

where

$$y''_{p,1} + 2y'_{p,1} + y_{p,1} = \frac{1}{t^3 e^t} \quad y''_{p,2} + 2y'_{p,2} + y_{p,2} = 3.$$

By the method of undetermined coefficients we have

$$y_{p,2}(t) = 3.$$

We find $y_{p,1}$ using the method of variation of parameters. $W(y_1, y_2) = e^{-2t}$ So

$$y_{p,1}(t) = -e^{-t} \int \frac{te^{-t}}{e^{-2t} t^3 e^t} dt + te^{-t} \int \frac{e^{-t}}{e^{-2t} t^3 e^t} dt =$$

$$- e^{-t} \int \frac{1}{t^2} dt + te^{-t} \int \frac{1}{t^3} dt = \frac{e^{-t}}{t} - \frac{e^{-t}}{2t}$$

Then

$$y(t) = \frac{e^{-t}}{2t} + 3 + C_1 e^{-t} + C_2 te^{-t}$$