QUIZ2 SPRING 2005

Problem 1. Find a solution to the initial value problem

$$y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

Solution.
The characteristic polynomial is $r^2 + 2r + 1 = (r+1)^2$. Roots of this polynomial are $r_1 = r_2 = -1$. The general solution is

$$y(t) = C_1 e^{-t} + C_2 t e^{-t}.$$

Next let's find constants C_1, C_2.

$$y(0) = 1 = C_1.$$

Hence

$$y(t) = e^{-t} + C_2 t e^{-t}.$$

Then

$$y'(t) = -e^{-t} + C_2 e^{-t} + C_2 t e^{-t}.$$

Therefore

$$y'(0) = 1 = -1 + C_2. \quad C_2 = 2$$

Solution to our problem given by formula

$$y(t) = e^{-t} + 2te^{-t}.$$

Problem 2. Find a general solution to the ordinary differential equation

$$y'' + y = tcos(t) + 1.$$

Solution The characteristic polynomial is $r^2 + 1$. Then roots are $r_{1,2} = \pm i$. The general solution to the homogeneous equation is

$$y_h(t) = C_1 cos(t) + C_2 sin(t).$$

We are looking for a particular solution in the form

$$y_p(t) = y_{p,1}(t) + y_{p,2}(t),$$

where

$$y''_{p,1} + y_{p,1} = tcos(t) \quad y''_{p,2} + y_{p,2} = 1$$

Note that $1 = e^{\gamma t}P_0(t)$ with $\gamma = 0$, $P_0(t) = 1$. Then $s = 0$ and we are looking for the function $y_{p,2}$ in the form $y_{p,2} = q_0$. Since $y''_{p,2} = 0$ we have

$$0 + q_0 = 1.$$

Hence $y_{p,2} = 1$.

Next we note that $tcos(t) = e^{\alpha t}(P_1(t)cos(\beta t) + R_1(t)sin(\beta t))$ with $\beta = 1, \alpha = 0, P_1(t) = t, R_1(t) = 0$. Since $\alpha + i\beta = i$ is a root to the characteristic equation we are looking for the function $y_{p,1}$ in the form

$$y_{p,1} = (q_0 t + q_1 t^2)cos(t) + (g_0 t + g_1 t^2)sin(t).$$

Note that

$$y''_{p,1} = 2q_1 cos(t) + 2g_1 sin(t) + 2(q_0 + 2q_1 t)(-sin(t)) + 2(g_0 + 2g_1 t)cos(t) - y_{p,1}$$

Then

$$2q_1 cos(t) + 2g_1 sin(t) + 2(q_0 + 2q_1 t)(-sin(t)) + 2(g_0 + 2g_1 t)cos(t) = tcos(t)$$

Hence

$$(2q_1 + 2g_0 + (4g_1 - 1)t)cos(t) + (2g_1 - 2q_0 - 4q_1 t)sin(t) = 0.$$

Hence we have a system of equation

$$2q_1 + 2g_0 = 0, \quad 4g_1 - 1 = 0 \quad 2g_1 - 2q_0 = 0 \quad -4q_1 = 0.$$

Hence

$$q_1 = 0,\ g_0 = -q_1 = 0,\ g_1 = \frac{1}{4},\ q_0 = g_1 = \frac{1}{4}.$$

Therefore

$$y_{p,2}(t) = \frac{t^2}{4}sin(t) + \frac{t}{4}cos(t).$$

Finally

$$y(t) = C_1 cos(t) + C_2 sin(t) + \frac{t^2}{4}sin(t) + \frac{t}{4}cos(t) + 1.$$

Problem 3. Find a general solution to the ordinary differential equation

$$y'' - 2y' + y = \frac{e^t}{1 + t^2}$$

Solution. Characteristic polynomial has one root $r_1 = r_2 = 1$. Then general solution to the homogeneous equation is

$$y_h(t) = c_1 e^t + c_2 t e^t.$$

We set $y_1(t) = e^t$ and $y_2(t) = te^t$. Then $W(t) = e^{2t}$. Using the formula on variation of parameters we obtain

$$y_p(t) = -e^t \int \frac{t}{1 + t^2}dt + te^t \int \frac{t}{1 + t^2}dt = -\frac{1}{2}e^t ln(1 + t^2) + te^t arctan(t).$$

Problem 4. For the ordinary differential equation

$$\frac{dy}{dt} = y^2(1 - e^y)(y - 1)$$

find all equilibrium solutions, and determine which of these equilibrium solutions are stable and which are unstable.

Solution. Solving the equation $f(y) = y^2(1 - e^y)(y - 1) = 0$ we have two solutions $y^* = 1$ and $y^* = 0$.

Near the point $y^* = 1$ the function $f(y)$ is decreasing so $y(t) = 1$ is the stable equilibrium solution; Near the point $y^* = 0$ the function $f(y)$ is increasing so $y(t) = 0$ is the unstable equilibrium solution;