QUIZ 1 (FALL 2003)

Problem 1. Find a general solution to the ordinary differential equation

$$(x^2 - y^2)\frac{dy}{dx} = -2xy. \tag{1}$$

*Solution.*This equation is exact equation. Really $M(x, y) = 2xy$ and $N(x, y) = x^2 - y^2$. So $M_y = 2x = N_x$. The function $F(x, y)$ given by formula $F(x, y) = x^2 y - \frac{y^3}{3}$. So if $y(x)$ is a solution to problem (1) then

$$x^2 y(x) - \frac{y^3(x)}{3} = C.$$

Problem 2. Find a general solution to the ordinary differential equation

$$\frac{dy}{dt} = \frac{y - t}{y + t}.$$

Solution. This equation is the homogeneous equation. Really if we take $F(z) = \frac{z-1}{z+1}$ we have

$$\frac{dy}{dt} = F(\frac{y}{t}).$$

Note that $F(z) - z = -\frac{z^2+1}{z+1}$. Hence for the function $v(t) = y(t)/t$ we have

$$t\frac{dv}{dt} = -\frac{v^2 + 1}{v + 1}$$

This equation is separable so we have

$$-\frac{1}{2}\ln(v^2(t) + 1) - arctan(v(t)) = \ln|t| + C$$

Problem 3. In 30 days 50% of the initial amount of the radioactive material decayed. Find the moment of time when only one percent of the initial amount of radioactive material is left.

Solution. Denote by $N(t)$ the amount of radioactive material at moment t. Then

$$\frac{dN}{dt} = -kN$$

And $N(t) = N_0 e^{-kt}$ where N_0 is the initial amount of radioactive material. Since $N(30) = \frac{N_0}{2}$ we have

$$\frac{N_0}{2} = N_0 e^{-30k}$$

So $k = \frac{\ln 2}{30}$ and

$$N(t) = N_0 e^{-\frac{\ln 2}{30} t} \tag{2}$$

Denote by t_0 the time moment when when only one percent of the initial amount of radioactive material is left. Hence $N(t_0) = N_0/100$. From (2) we have

$$N_0/100 = N_0 e^{-\frac{\ln 2}{30} t_0}$$

and

$$t_0 = \frac{30 \ln 100}{\ln 2}.$$

Problem 4. The tank of conic form has a plug on the bottom. The water level at the tank is $36ft$. At 1PM the plug was removed and at 2PM the tank was empty. Find the time moment when the water level in the tank is $16ft$.

Solution. Denote by $y(t)$ the water level in the tank at moment t. Then $y(1) = 36$ and $y(2) = 0$ By Torricelli's law

$$y^2 \frac{dy}{dt} = -k\sqrt{y}$$

So

$$y^{\frac{3}{2}} \frac{dy}{dt} = -k \tag{3}$$

and

$$y^{\frac{5}{2}}(t) = -kt + C$$

Since $y(1) = 36$ we have $y^{\frac{5}{2}}(1) = 6^5 = -k + C$. Thus $C = k + 6^5$ and the formula (3) has the form

$$y^{\frac{5}{2}}(t) = -k(t-1) + 6^5.$$

Then

$$y^{\frac{5}{2}}(2) = 0 = -k + 6^5.$$

Hence $k = 6^5$ and

$$y^{\frac{5}{2}}(t) = -6^5(t-1) + 6^5.$$

Let t_0 be the moment of time such that $y(t_0) = 16$ We have

$$4^5 = -6^5(t_0 - 1) + 6^5.$$

So

$$t_0 = 2 - \frac{4^5}{6^5}.$$