SOLUTIONS FOR A SAMPLE EXAM.

Problem 1. We use the divergence theorem. Note that $div\mathbf{F} = 2x$. Then

$$\int\int_S \mathbf{F}\cdot dS = -\int\int\int_E div\,\mathbf{F}dV = -\int\int\int_E 2xdV$$
$$= -\int\int_D \int_0^{1-x^2-y^2} 2xdzdA.$$

Here $D = \{(x,y)|x^2+y^2 \leq 1\}$.

$$\int\int_S \mathbf{F}\cdot dS = -\int\int_D (2x-2x^3-2xy^2)dA = -\int_0^{2\pi}\int_0^1 (2rcos\theta-2r^3cos^3\theta-2r^2cos\theta sin\theta)rdrd\theta$$
$$-\int_0^{2\pi}(r^2cos\theta-\frac{r^4}{2}sin^3\theta-\frac{r^4}{2}cos\theta sin^2\theta)|_0^1 d\theta = -\int_0^{2\pi}(cos\theta-\frac{sin^3\theta}{2}-\frac{1}{2}cos\theta(1-cos^2\theta))d\theta = 0.$$

Problem 2. We parameterize the line segment $[A,B]$ in the following way

$$r(t) = t\vec{i} + 4t\vec{j} + 3t\vec{k}, \quad 0 \leq t \leq 1.$$

Then

$$\int_C (2xy+sin(x))dx+x^2zdy+x^2ydz = \int_0^1 (2t\cdot 4t\cdot 1+sin(t)\cdot 1+t^2\cdot 3t\cdot 4+t^2\cdot 4t\cdot 3)dt =$$
$$\int_0^1 (8t^2+12t^3+12t^3+sin(t))dt = \int_0^1 (8t^2+24t^3+sin(t))dt = (\frac{8}{3}t^3+6t^4-cos(t))|_0^1$$
$$= \frac{8}{3}+6-cos(1)+1.$$

Problem 3. By the Stokes' Theorem

$$\int_C \mathbf{F}\cdot dr = \int\int_S curl\mathbf{F}dS.$$

We have

$$curl\mathbf{F} = \vec{i} - (2x+1)\vec{j}.$$

The surface is given as a graph of function

$$z = g(x,y) = 6 - \frac{3x}{2} - \frac{y}{2}.$$

Obviously

$$g_x = -\frac{3}{2}, \quad g_y = -\frac{1}{2}.$$

Computing the dot product

$$(\vec{i} - (2x+1)\vec{j}) \cdot (\frac{3}{2}\vec{i} + \frac{1}{2}\vec{j} + \vec{k}) = 1 - x$$

Hence

$$\int_C \mathbf{F} \cdot dr = \int\int_S curl\mathbf{F} dS = -\int\int_R (1-x)dA = -\int_0^4 \int_0^{12-3x} (1-x)dydx =$$
$$-\int_0^4 (1-x)(12-3x)dx = -\int_0^4 (12-12x-3x+3x^2)dx = -\int_0^4 (12-15x+3x^2)dx =$$
$$-(12x - \frac{15x^2}{2} + x^3)|_0^4 = -48 + 120 - 64 = -8.$$

Problem 4. Note that the vector field $\mathbf{F} = xy^2\vec{i} + x^2y\vec{j}$ is conservative. Really

$$P_y = 2xy = Q_x = 2xy.$$

Therefore since the curve is closed $r(0) = r(2\pi)$, by the fundamental theorem of the line integrals

$$\int_C \mathbf{F} \cdot dr = 0.$$