LECTURE 20 (267)

Endpoint problems.

By the endpoint problem for the second order linear ordinary differential equation we mean the following problem. Let four constants c_1, c_2, c_3, c_4 be given. Assume that $(c_1, c_2) \neq 0$ and $(c_3, c_4) \neq 0$. We are looking for a function $y(t)$ such that this function satisfies to the ordinary differential equation

$$y'' + p(t)y' + q(t)y = f(t) \quad t \in [a, b], \tag{1}$$

and the **boundary conditions**

$$c_1 y'(a) + c_2 y(a) = y_0, \quad c_3 y'(b) + c_4 y(b) = y_1. \tag{2}$$

The difference between *initial value problem* and *end point problem* is the following. For *initial value problem* we fix values of y' and y at the single point $t = a$. For the *boundary value problem* we have one condition on the linear combination of y', y at point $t = a$ and another at $t = b$. In general for given y_0, y_1 and continuous function f solution to the endpoint problem may not exist. It might happen that even if a solution to the endpoint problem exists it is not unique.

Definition. *We say that the function $f(t)$ is not equal identically zero on the interval $[a, b]$ if there exists a point $\tau_0 \in [a, b]$ such that $f(\tau_0) \neq 0$.*

Definition. *A number λ is called eigenvalue and a function $y(t)$ is called eigenfunction which corresponds to this eigenvalue if*

$$y'' + p(t)y' + q(t)y + \lambda y = 0 \quad t \in [a, b], \tag{3}$$

and the **boundary conditions**

$$c_1 y'(a) + c_2 y(a) = 0, \quad c_3 y'(b) + c_4 y(b) = 0. \tag{4}$$

Example 1. Find eigenvalues and eigenfunctions:

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0.$$

Solution. We consider three cases.
Case 1. Let $\lambda = 0$. The characteristic equation is

$$r^2 = 0$$

We have only one root $r = 0$ of multiplicity two. So the general solution is

$$y(t) = c_1 + c_2 t$$

Since $y'(0) = 0$ we have $c_2 = 0$ So

$$y(t) = c_1$$

Since $y(1) = 0$ then $c_1 = 0$ and $y(t) = 0$. Hence $\lambda = 0$ is not the eigenvalue.
Case 2. Let $\lambda < 0$. The characteristic equation is

$$r^2 = -|\lambda|.$$

We have two roots $r = \pm\sqrt{|\lambda|}$ of multiplicity one. So the general solution is

$$y(t) = c_1 e^{\sqrt{|\lambda|}t} + c_2 e^{-\sqrt{|\lambda|}t}.$$

Since $y'(0) = 0$ we have $c_2 = c_1$. So

$$y(t) = c_1 e^{\sqrt{|\lambda|}t} + c_1 e^{-\sqrt{|\lambda|}t}.$$

Since $y(1) = 0$ we have

$$y(1) = 0 = c_1 e^{\sqrt{|\lambda|}} + c_1 e^{-\sqrt{|\lambda|}}.$$

This imply $c_1 = 0$ and $y(t) = 0$. Hence for $\lambda < 0$ there are no eigenvalues.
Case 3. Let $\lambda > 0$. The characteristic equation is

$$r^2 = -\lambda.$$

We have two complex roots $r = \pm\sqrt{\lambda}i$ of multiplicity one. So the general solution is

$$y(t) = c_1 cos(\sqrt{\lambda}t) + c_2 sin(\sqrt{\lambda}t).$$

Note that

$$y'(t) = -c_1\sqrt{\lambda}sin(\sqrt{\lambda}t) + c_2\sqrt{\lambda}cos(\sqrt{\lambda}t).$$

Since $y'(0) = 0$ we have $c_2 = 0$ and the possible eigenfunction has a form

$$y(t) = c_1 cos(\sqrt{\lambda}t).$$

We should have

$$y(1) = 0 = c_1 cos(\sqrt{\lambda}).$$

we can not take $c_1 = 0$, so we need to solve the equation

$$cos(\sqrt{\lambda}) = 0$$

In that case we have

$$\sqrt{\lambda} = \frac{\pi}{2} + \pi k$$

where k is the integer non negative number. So the eigenvalues are

$$\lambda = (\frac{\pi}{2} + \pi k)^2 \quad \text{where } k \text{ is the integer non negative number.}$$

and the eigenfunctions are

$$y(t) = Ccos((\frac{\pi}{2}+\pi k)t) \quad \text{where } k \text{ is the integer non negative number and } c \neq 0.$$

Example 2. Find eigenvalues and eigenfunctions

$$y'' + \lambda y = 0, \quad y'(-\pi) = 0, \quad y'(\pi) = 0.$$

Solution. We consider three cases.
Case 1. Let $\lambda = 0$. The characteristic equation is

$$r^2 = 0$$

We have only one root $r = 0$ of multiplicity two. So the general solution is

$$y(t) = c_1 + c_2 t$$

Since $y'(-\pi) = 0$ we have $c_2 = 0$ So

$$y(t) = c_1$$

Obviously $y'(\pi) = 0$. Hence $\lambda = 0$ is the eigenvalue and $y(t) = c_1$ is the eigenfunction if $c_1 \neq 0$.

Case 2. Let $\lambda < 0$. The characteristic equation is

$$r^2 = -|\lambda|.$$

We have two roots $r = \pm\sqrt{|\lambda|}$ of multiplicity one. So the general solution is

$$y(t) = c_1 e^{\sqrt{|\lambda|}t} + c_2 e^{-\sqrt{|\lambda|}t}.$$

Since $y'(-\pi) = 0$ we have

$$c_1 e^{-\sqrt{|\lambda|}\pi} + c_2 e^{\sqrt{|\lambda|}\pi} = 0. \tag{5}$$

Since $y'(\pi) = 0$ we have

$$y'(\pi) = 0 = c_1\sqrt{|\lambda|}\pi e^{\sqrt{|\lambda|}\pi} + c_2\sqrt{|\lambda|}\pi e^{-\sqrt{|\lambda|}\pi}. \tag{6}$$

From equation (5) we have

$$c_1 e^{\sqrt{|\lambda|}\pi} = -c_2 e^{3\sqrt{|\lambda|}\pi} \tag{7}$$

So we can transform the equation (6) as

$$-c_2 e^{3\sqrt{|\lambda|}\pi} + c_2\sqrt{|\lambda|}\pi e^{-\sqrt{|\lambda|}\pi} = 0. \tag{8}$$

From (8) we obtain $c_2 = 0$. Then (7) imply $c_1 = 0$ and $y(t) = 0$. Hence for $\lambda < 0$ there are no eigenvalues.

Case 3. Let $\lambda > 0$. The characteristic equation is

$$r^2 = -\lambda.$$

We have two complex roots $r = \pm\sqrt{\lambda}i$ of multiplicity one. So the general solution is

$$y(t) = c_1 cos(\sqrt{\lambda}t) + c_2 sin(\sqrt{\lambda}t).$$

Note that

$$y'(t) = -c_1\sqrt{\lambda}sin(\sqrt{\lambda}t) + c_2\sqrt{\lambda}cos(\sqrt{\lambda}t).$$

Since $y'(-\pi) = y(\pi) = 0$ we have the system of linear equations on unknowns c_1, c_2:

$$\begin{cases} c_1\sqrt{\lambda}sin(\sqrt{\lambda}\pi) + c_2\sqrt{\lambda}cos(\sqrt{\lambda}\pi) = 0 \\ -c_1\sqrt{\lambda}sin(\sqrt{\lambda}\pi) + c_2\sqrt{\lambda}cos(\sqrt{\lambda}\pi) = 0 \end{cases} \tag{9}$$

We are looking for nonzero solutions to the linear system (9). This system has nonzero solution if and only if the determinant of the following matrix is zero:

$$\begin{pmatrix} \sqrt{\lambda}sin(\sqrt{\lambda}\pi) & \sqrt{\lambda}cos(\sqrt{\lambda}\pi) \\ -\sqrt{\lambda}sin(\sqrt{\lambda}\pi) & \sqrt{\lambda}cos(\sqrt{\lambda}\pi) \end{pmatrix}.$$

The determinant of this matrix is

$$\lambda sin(\sqrt{\lambda}\pi)cos(\sqrt{\lambda}\pi) = \frac{\lambda}{2}sin(2\sqrt{\lambda}\pi).$$

So we need to solve the equation

$$sin(2\sqrt{\lambda}\pi) = 0$$

In that case we have

$$\sqrt{\lambda} = \frac{k}{2}$$

where k is the integer non negative number. So the eigenvalues are

$$\lambda = \frac{k^2}{4} \quad \text{where } k \text{ is the integer non negative number.}$$

Now we find the eigenfunctions which corresponds to the eigenvalues. They should have a form

$$y(t) = c_1cos(\frac{kt}{2}) + c_2sin(\frac{kt}{2}).$$

and

$$y'(t) = -\frac{c_1k}{2}sin(\frac{kt}{2}) + \frac{c_2k}{2}sin(\frac{kt}{2}).$$

Since $y'(\pi) = y'(-\pi) = 0$ we have $c_2 = 0$. So the eigenfunctions have the form

$$y(t) = c_1cos(\frac{kt}{2})$$

where $c_1 \neq 0$.