LECTURE 16 (267)

Variation of Parameters.

We consider the second order linear differential equation

$$y'' + P(t)y' + Q(t)y = f(t), \tag{1}$$

where $P(t), Q(t)$ and $f(t)$ are continuous functions. Suppose we know the general solution to the homogeneous equation

$$y'' + P(t)y' + Q(t)y = 0$$

We denote this general solution as

$$Y(t) = c_1 y_1(t) + c_2 y_2(t).$$

We are looking for a particular solution $y_p(t)$ to equation (1). We have

$$y_p(t) = -y_1(t)\int \frac{y_2(t)f(t)}{W(t)}dt + y_2(t)\int \frac{y_1(t)f(t)}{W(t)}dt. \tag{2}$$

Here $W(t)$ is the Wronskian of two functions $y_1(t), y_2(t)$.

$$W(t) = W(y_1, y_2) = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Example 1. Find a general solution to equation

$$y'' + y = \frac{1}{sin(t)}.$$

Solution. The characteristic polynomial is

$$r^2 + 1.$$

We have two complex roots $r_{1,2} = \pm i$. Therefore $y_1(t) = cost$ and $y_2(t) = sin(t)$. Hence $W(t) = 1$. Applying the formula (2) we have

$$y_p(t) = -cost \int 1dt + sint \int \frac{cost}{sint} dt = -tcost + \ln|sint|.$$

The general solution is

$$y(t) = c_1 cost + c_2 sint - tcost + \ln|sint|.$$

Example 2. Find a general solution to equation

$$y'' + 3y' + 2y = \frac{1}{e^t + 1}.$$

Solution. The characteristic polynomial is

$$r^2 + 3r + 2.$$

We have two real roots $r_1 = -1, \quad r_2 = -2$. Therefore $y_1(t) = e^{-t}$ and $y_2(t) = e^{-2t}$. Hence $W(t) = -e^{-3t}$. Applying the formula (2) we have

$$\begin{aligned} y_p(t) = e^{-t} \int \frac{e^t}{e^t+1} dt - e^{-2t} \int \frac{e^{2t}}{e^t+1} dt = \\ e^{-t}\ln(e^t+1) - e^{-2t} \int \frac{e^{2t}+e^t-e^t}{e^t+1} dt = \\ e^{-t}\ln(e^t+1) - e^{-2t} \int (e^t + \frac{-e^t}{e^t+1}) dt = \\ e^{-t}\ln(e^t+1) - e^{-t} + e^{-2t}\ln(e^t+1). \end{aligned}$$

The general solution is

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + e^{-t}\ln(e^t+1) - e^{-t} + e^{-2t}\ln(e^t+1).$$

Example 3. Find a general solution to equation

$$y'' + 2y' + y = 3e^{-t}\sqrt{t+1}.$$

Solution. The characteristic polynomial is

$$r^2 + 2r + 1.$$

We have one real root of multiplicity 2 $r_{1,2} = 1$. Therefore $y_1(t) = e^t$ and $y_2(t) = te^t$. Hence $W(t) = e^{2t}$. Applying the formula (2) we have

$$y_p(t) = -e^t \int 3t\sqrt{t+1}dt + te^t \int 3\sqrt{t+1}dt =$$

$$-e^t \int 3((t+1)-1)\sqrt{t+1}dt + te^t \int 3\sqrt{t+1}dt = -e^t 3(\frac{2}{5}(t+1)^{\frac{5}{2}} - \frac{2}{3}(t+1)^{\frac{3}{2}}) + te^t 2(t+1)^{\frac{3}{2}}$$

the general solution is

$$y_p(t) = c_1 e^t + c_2 te^t - e^t 3(\frac{2}{5}(t+1)^{\frac{5}{2}} - \frac{2}{3}(t+1)^{\frac{3}{2}}) + te^t 2(t+1)^{\frac{3}{2}}.$$