APPLICATIONS OF SYSTEMS OF DIFFERENTIAL EQUATIONS.

Example 1. Consider two tanks filled with brine connected by pipes. Through the first pipe the brine is coming from the first tank to the second tank at the rate $r = 10gal/min$. Through the second pipe brine coming from the second tank to the first tank at the same rate $r = 10gal/min$. Initially volume of brine in the first tank is $50gal$ and amount of salt is $15lb$. The second tank initially contains $25gal$ of pure water. Find amount of salt in the first and second tanks at moment t.

Solution. Let $x_1(t)$ be amount of salt in the first tank and $x_2(t)$ be amount of salt into the second tank. Since the water is coming on rate $10gal/min$ to each tank and leaving tank at the same rate the volume of brine is constant: $V_1 = 50$ and $V_2 = 25$. The functions x_1, x_2 satisfy to the system of the ordinary differential equations

$$x_1' = -\frac{1}{5}x_1 + \frac{2}{5}x_2, \quad x_2' = \frac{1}{5}x_1 - \frac{2}{5}x_2, \quad x_1(0) = 15,\ x_2(0) = 0. \tag{1}$$

Therefore we have

$$A = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{pmatrix}$$

First we find an eigenvalues of the matrix A:

$$A - \lambda E = \begin{pmatrix} -\frac{1}{5} - \lambda & \frac{2}{5} \\ \frac{1}{5} & -\frac{2}{5} - \lambda \end{pmatrix}.$$

Then $det(A - \lambda E) = (-\frac{1}{5} - \lambda)(-\frac{2}{5} - \lambda) - \frac{2}{25}$. We have two eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -\frac{3}{5}$. Let find an eigenvector associated with the eigenvalue $\lambda_1 = 0$. The system $A\vec{e} = 0$ equivalent to the equation $-e_1 + 2e_2 = 0$. So $e_1 = 2$ and $e_2 = 1$. Let find an eigenvector associated with the eigenvalue $\lambda_2 = -\frac{3}{5}$. We have

$$A + \frac{3}{5}E = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}.$$

So the system $(A+\frac{3}{5}E)\vec{e}=0$ is equivalent to the equation $e_1+e_2=0$. Hence $e_1=-1$ and $e_2=1$. the general solution to the system (1) is

$$x(t)=C_1\begin{pmatrix}2\\1\end{pmatrix}+C_2\begin{pmatrix}2\\1\end{pmatrix}e^{-\frac{3}{5}t}.$$

Using the initial condition we find the constants C_1 and C_2. We have

$$\begin{pmatrix}15\\0\end{pmatrix}=C_1\begin{pmatrix}2\\1\end{pmatrix}+C_2\begin{pmatrix}2\\1\end{pmatrix}.$$

Then $C_1=5$ and $C_2=5$. Hence the answer is

$$x(t)=5\begin{pmatrix}2\\1\end{pmatrix}+5\begin{pmatrix}2\\1\end{pmatrix}e^{-\frac{3}{5}t}.$$

Example 2. A pipe connected to the tank 1. Through this pipe the brine is coming to the tank at the rate $r=20gal/min$ with the concentration of the salt $c=2$. The tan 1 connected to the tank 2 by pipe. Through this pipe the water is coming at the rate $r=20gal/min$ and leaving this tank at the same rate. Initially the tanks contain $100gal$ and $200gal$ of pure water. Find the limiting amount of salt in both tanks as time goes to infinity. Find moments of time when concentration of salt reach $1lb/gal$ in the tanks.

Solution. Let $x_1(t)$ be amount of salt in the first tank and $x_2(t)$ be amount of salt into the second tank. Let $c_1(t)$ be amount of salt in the first tank and $c_2(t)$ be concentration of salt into the second tank. It is known that

$$C_i(t)=x_i(t)/V_i.$$

Since the water is coming on rate $10gal/min$ to each tank and leaving tank at the same rate the volume of brine is constant: $V_1=100$ and $V_2=200$. The functions x_1,x_2 satisfy to the system of the ordinary differential equations

$$x_1'=-\frac{1}{10}x_1+20,\quad x_2'=\frac{1}{10}x_1-\frac{1}{20}x_2,\quad x_1(0)=0,\ x_2(0)=0. \tag{2}$$

Solving the equation for the function $x_1(t)$ we have

$$x_1(t)=C_1e^{-\frac{t}{10}}+200.$$

Using the initial condition $x_1(0)=0$ we have

$$x_1(t)=200(1-e^{-\frac{t}{10}}). \tag{3}$$

c Then

$$c_1(t) = \frac{x_1(t)}{V_1} = 2(1 - e^{-\frac{t}{10}}).$$

Denote by τ_1 the moment of time when $c_1(\tau_1) = 1$. Then

$$c_1(\tau_1) = 1 = 2(1 - e^{-\frac{\tau_1}{10}}).$$

Hence $\tau_1 = 10\ln(2)$. Using (2),(3) we obtain the equation for the function $x_2(t)$:

$$x_2' = 20(1 - e^{-\frac{t}{10}}) - \frac{1}{20}x_2.$$

The

$$x_2(t) = C_1 e^{-\frac{t}{20}} + 400 + 400e^{-\frac{t}{10}}.$$

Then $C_1 = -800$ and

$$x_2(t) = -800e^{-\frac{t}{20}} + 400 + 400e^{-\frac{t}{10}}.$$

note that

$$\lim_{t\to+\infty} x_2(t) = 400.$$