LECTURE 41 (267)

Nonhomogeneous systems of O.D.E.

Example 1. Find a general solution to the system of the ordinary differential equations

$$x' = 4x + y + e^t, \quad y' = 6x - y - e^t.$$

Solution. We set $X(t) = (x(t), y(t))$, $f(t) = (e^t, -e^t)$, and

$$A = \begin{pmatrix} 4 & 1 \\ 6 & -1 \end{pmatrix}.$$

Then we can rewrite the system in the form

$$\frac{dX}{dt} = AX + f.$$

Lets find eigenvalues and eigenvectors of the matrix A we have

$$A - \lambda E = \begin{pmatrix} 4-\lambda & 1 \\ 6 & -1-\lambda \end{pmatrix}.$$

Then $det(A-\lambda E) = (4-\lambda)(-1-\lambda)-6 = -4-4\lambda+\lambda+\lambda^2-6 = \lambda^2-3\lambda-10$. So we have $\lambda_1 = -2$ and $\lambda_2 = 5$. Lets find an eigenvector associated with the eigenvalue $\lambda_1 = -2$. We have

$$A + 2E = \begin{pmatrix} 6 & 1 \\ 6 & 1 \end{pmatrix}.$$

So the system $(A+2E)\vec{e} = 0$ equivalent to the equation $6e_1 + e_2 = 0$. Hence our eigenvector is $e = (1, -6)$. Therefore the solution associated with this eigenvalue is $x_1(t) = e^{-2t}(1, -6)$. Lets find an eigenvector associated with the eigenvalue $\lambda_2 = 5$. We have

$$A - 5E = \begin{pmatrix} -1 & 1 \\ 6 & -6 \end{pmatrix}.$$

So the system $(A - 5E)\vec{e} = 0$ equivalent to the equation $e_1 - e_2 = 0$. Hence our eigenvector is $e = (3, 2)$. Therefore the solution associated with this eigenvalue is $x_1(t) = e^{5t}(1, 1)$. Hence the fundamental matrix

$$\Phi(t) = \begin{pmatrix} e^{-2t} & e^{5t} \\ -6e^{-2t} & e^{5t} \end{pmatrix}.$$

Next we find the matrix $\Phi^{-1}(t)$. Note that $det\Phi(t) = 7e^{3t}$. Then

$$\Phi^{-1}(t) = \frac{1}{7e^{3t}} \begin{pmatrix} e^{5t} & -e^{5t} \\ 6e^{-2t} & e^{-2t} \end{pmatrix} = \frac{1}{7} \begin{pmatrix} e^{2t} & -e^{2t} \\ 6e^{-5t} & e^{-5t} \end{pmatrix}.$$

Then we have

$$\Phi^{-1}(t)f(t) = \frac{1}{7}(2e^{3t}, 5e^{-4t}), \quad \int \Phi^{-1}(t)f(t)dt = \frac{1}{7}(\frac{2}{3}e^{3t}, -\frac{5}{4}e^{-4t}).$$

Finally

$$x_p(t) = \Phi(t) \int \Phi^{-1}(t)f(t)dt = (-\frac{1}{12}, \frac{3}{4})).$$

The general solution is

$$X(t) = C_1 e^{-2t} \begin{pmatrix} 1 \\ -6 \end{pmatrix} + C_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{12} \\ \frac{3}{4} \end{pmatrix}.$$

Example 2. Find a general solution to the system of the ordinary differential equations

$$x' = x - 5y + cos(2t), \quad y' = x - y.$$

Solution.

We set $X(t) = (x(t), y(t))$, $f(t) = (cos(2t), 0)$, and

$$A = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}.$$

Then we can rewrite the system in the form

$$\frac{dX}{dt} = AX + f.$$

Lets find eigenvalues and eigenvectors of the matrix A we have

$$A - \lambda E = \begin{pmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{pmatrix}.$$

Then $det(A - \lambda E) = (1-\lambda)(-1-\lambda) + 5 = \lambda^2 + 4$. So we have $\lambda_1 = 2i$ and $\lambda_2 = -2i$. Lets find an eigenvector associated with the eigenvalue $\lambda_1 = 2i$. We have

$$A - 2iE = \begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix}.$$

So the system $(A - 2iE)\vec{e} = 0$ equivalent to the equation $e_1 - (1+2i)e_2 = 0$. Hence our eigenvector is $e = (1+2i, 1)$. Therefore the solutions associated with this eigenvalues $\pm 2i$ are

$$x_1(t) = \begin{pmatrix} cos(2t) - 2sin(2t) \\ cos(2t) \end{pmatrix}, \quad x_2(t) = \begin{pmatrix} 2cos(2t) + sin(2t) \\ sin(2t) \end{pmatrix}.$$

Hence the fundamental matrix

$$\Phi(t) = \begin{pmatrix} cos(2t) - 2sin(2t) & 2cos(2t) + sin(2t) \\ cos(2t) & sin(2t) \end{pmatrix}.$$

Next we find the matrix $\Phi^{-1}(t)$. Note that $det\Phi(t) = -2$. Then

$$\Phi^{-1}(t) = \frac{1}{-2} \begin{pmatrix} sin(2t) & -2cos(2t) - sin(2t) \\ -cos(2t) & cos(2t) - 2sin(2t) \end{pmatrix}.$$

Then we have

$$\Phi^{-1}(t)f(t) = \frac{1}{-2}(\frac{1}{2}sin(4t), -cos^2(2t)), \quad \int \Phi^{-1}(t)f(t)dt = \frac{1}{2}(\frac{1}{8}cos(4t), \frac{1}{2}(t+\frac{1}{4}sin(2t))).$$

Finally

$$x_p(t) = \Phi(t)\int \Phi^{-1}(t)f(t)dt = \begin{pmatrix} (cos(2t) - 2sin(2t))\frac{1}{16}cos(4t) + (2cos(2t) + sin(2t))\frac{1}{4}(t + \frac{1}{4}sin(2t)) \\ \frac{1}{16}cos(4t)cos(2t) + \frac{1}{4}sin(2t)(t + \frac{1}{4}sin(2t)) \end{pmatrix}$$

The general solution is

$$X(t) = C_1 \begin{pmatrix} 2cos(2t) + sin(2t) \\ sin(2t) \end{pmatrix} + C_2 \begin{pmatrix} cos(2t) - 2sin(2t) \\ cos(2t) \end{pmatrix} +$$
$$\begin{pmatrix} (cos(2t) - 2sin(2t))\frac{1}{16}cos(4t) + (2cos(2t) + sin(2t))\frac{1}{4}(t + \frac{1}{4}sin(2t)) \\ \frac{1}{16}cos(4t)cos(2t) + \frac{1}{4}sin(2t)(t + \frac{1}{4}sin(2t)) \end{pmatrix}$$