

LECTURE 43 (267)

Homogeneous systems of linear ordinary differential equations, function e^{At} .

Let A be the matrix of the size $n \times n$. We introduce the matrix e^{At} by formula

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}.$$

The elements of the matrix e^{At} are t -dependent functions. The matrix e^{At} has the following property

$$\frac{d}{dt} e^{At} = A e^{At}.$$

Using the matrix e^{At} we can get a simple formula for solutions of the initial value problems. We have the following theorems

Theorem 1. *Consider the initial value problem for the homogeneous system*

$$\frac{dX}{dt} = AX, \quad X(0) = x_0. \quad (1)$$

Then solution to the initial value problem (1) given by formula

$$X(t) = e^{At} x_0. \quad (2)$$

Next we consider non homogeneous system

Theorem 2.. *Consider the initial value problem for the homogeneous system*

$$\frac{dX}{dt} = AX + f(t), \quad X(0) = x_0. \quad (3)$$

then solution to the initial value problem (3) given by formula

$$X(t) = e^{At} x_0 + e^{At} \int_0^t e^{-As} f(s) ds. \quad (4)$$

The formulae (2),(4) are nice but it is not simple to compute the matrix e^{At} . If $\Phi(t)$ is the fundamental matrix to the homogeneous system $\frac{dX}{dt} = AX$, then $e^{At} = \Phi(t)\Phi^{-1}(0)$. In some situations the infinite series $\sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$ contains only finite number of non zero terms.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

Definition. A matrix A is called nilpotent if for some integer k $A^k = 0$.

Example 1. Find e^{At} for the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Solution. First we find the matrix A^2 . We have

$$A^2 = A \cdot A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence the matrix A is nilpotent with $k = 2$. Therefore

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^1 \frac{A^n t^n}{n!} = E + At = \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix}$$

Example 2 Find e^{At} for the matrix

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution. First we find the matrix A^2 . We have

$$A^2 = A \cdot A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore the matrix A is nilpotent with $k = 3$. Hence

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^2 \frac{A^n t^n}{n!} = E + At + \frac{A^2 t^2}{2} = \begin{pmatrix} 1+t & -t-t^2 & -t \\ t & 1-t & t-t^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The systems of ordinary differential equations of order higher than one can be reduced to the first order system of O.D.E.. We consider two examples which illustrate the general algorithm.

Let's consider the second order O.D.E.

$$y'' + p(t)y' + q(t)y = f(t). \quad (5)$$

We introduce the function $x(t) = (x_1(t), x_2(t))$ such that $x_1(t) = y(t)$, $x_2(t) = y'(t)$. Then obviously we have $x'_1 = x_2$. The equation (5) one can rewrite as

$$x'_2 + p(t)x_2 + q(t)x_1 = f(t)$$

Then we have

$$x' = Ax + F(t),$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \quad F(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Next we consider the system of two second order equations

$$y'' + p_1(t)y' + p_2(t)z' + q_1(t)y + q_2(t)z = f_1(t), \quad z'' + p_3(t)y' + p_4(t)z' + q_3(t)y + q_4(t)z = f_2(t)$$

We set $X(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ where $x_1(t) = y(t)$, $x_2(t) = y'(t)$, $x_3(t) = z(t)$, $x_4(t) = z'(t)$. Then

$$x'_1 = x_2, \quad x'_3 = x_4$$

and

$$x'_2 + p_1(t)x_2 + p_2(t)x_4 + q_1(t)x_1 + q_2(t)x_3 = f_1(t), \quad x'_4 + p_3(t)x_2 + p_4(t)x_4 + q_3(t)x_1 + q_4(t)x_3 = f_2(t)$$

Then we have

$$x' = Ax + F(t),$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -q_1(t) & -p_1(t) & -q_2(t) & -p_2(t) \\ 0 & 0 & 0 & 1 \\ -q_3(t) & -p_3(t) & -q_4(t) & -p_4(t) \end{pmatrix} \quad F(t) = \begin{pmatrix} 0 \\ f_1 \\ 0 \\ f_2 \end{pmatrix}.$$