LECTURE 41 (267)

Nonhomogeneous systems of O.D.E.

We consider the system

$$\frac{dX}{dt} = P(t)X + f(t), \tag{1}$$

Here $f(t) = (f_1(t), \dots, f_n(t))$ ia a given vector function, and $X(t) = (x_1, \dots, x_n)$ is unknown vector function. The general solution to the system (1) has the form

$$X(t) = x_c(t) + x_p(t), \tag{2}$$

where $x_c(t)$ is a general solution to the homogeneous system

$$\frac{dx_c}{dt} = P(t)x_c \tag{3}$$

and x_p is a particular solution to the system (1):

$$\frac{dx_p}{dt} = P(t)x_p + f(t), \tag{4}$$

Let $x_c(t) = C_1x_1(t)+C_2x_2(t)+\dots C_nx_n(t)$, where the functions $x_1, x_2, \dots x_n$ are the linearly independent solutions to the homogeneous system (3). Then by the fundamental matrix $\Phi(t)$ we mean the following matrix

$$\Phi(t) = (x_1(t), x_2(t), \dots, x_n(t)).$$

Next we introduce the definition of the inverse matrix. If A is a matrix of the size $n \times n$ then where exists and unique a matrix, which we denote as A^{-1}, with the following properties

$$A^{-1}A = AA^{-1} = E.$$

If the matrix A has the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the inverse matrix has the form

$$A^{-1} = \frac{1}{detA} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

the particular solution given by the following formula

$$x_p(t) = \Phi(t) \int \Phi^{-1}(t) f(t) dt.$$

Example 1 Find a general solution to the system of the ordinary differential equations

$$x' = x + 2y + 3, \quad y' = 2x + y - 2.$$

Solution. We set $X(t) = (x(t), y(t))$, $f(t) = (3, -2)$, and

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then we can rewrite the system in the form

$$\frac{dX}{dt} = AX + f.$$

Lets find eigenvalues and eigenvectors of the matrix A we have

$$A - \lambda E = \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}.$$

Then $det(A - \lambda E) = (\lambda - 1)^2 - 4$. So we have $\lambda_1 = -1$ and $\lambda_2 = 3$. Lets find an eigenvector associated with the eigenvalue $\lambda_1 = -1$. We have

$$A + E = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

So the system $(A + E)\vec{e} = 0$ equivalent to the equation $e_1 + e_2 = 0$. Hence our eigenvector is $e = (1, -1)$. Therefore the solution associated with this eigenvalue is $x_1(t) = e^{-t}(1, -1)$. Lets find an eigenvector associated with the eigenvalue $\lambda_2 = 3$. We have

$$A - 3E = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}.$$

So the system $(A - 3E)\vec{e} = 0$ equivalent to the equation $e_1 - e_2 = 0$. Hence our eigenvector is $e = (1, 1)$. Therefore the solution associated with this eigenvalue is $x_1(t) = e^{3t}(1, 1)$. Hence the fundamental matrix

$$\Phi(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix}.$$

Next we find the matrix $\Phi^{-1}(t)$. Note that $det\Phi(t) = 2e^{2t}$. Then

$$\Phi^{-1}(t) = \frac{1}{2e^{2t}} \begin{pmatrix} e^{3t} & -e^{3t} \\ e^{-t} & e^{-t} \end{pmatrix}.$$

Then we have

$$\Phi^{-1}(t)f(t) = \frac{1}{2}(5e^t, e^{-3t}), \quad \int \Phi^{-1}(t)f(t)dt = \frac{1}{2}(5e^t, -\frac{1}{3}e^{-3t})$$

Finally

$$x_p(t) = \Phi(t) \int \Phi^{-1}(t)f(t)dt = (\frac{7}{3}, -\frac{8}{3}).$$

The general solution is

$$X(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{7}{3} \\ -\frac{8}{3} \end{pmatrix}.$$

Example 2. Find a general solution to the system of the ordinary differential equations

$$x' = 2x + 3y + 5, \quad y' = 2x + y - 2t.$$

Solution. We set $X(t) = (x(t), y(t))$, $f(t) = (5, -2t)$, and

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}.$$

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Then we can rewrite the system in the form

$$\frac{dX}{dt} = AX + f.$$

Lets find eigenvalues and eigenvectors of the matrix A we have

$$A - \lambda E = \begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix}.$$

Then $det(A - \lambda E) = (2-\lambda)(1-\lambda) - 6 = 2 - 2\lambda - \lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda - 4$. So we have $\lambda_1 = -1$ and $\lambda_2 = 4$. Lets find an eigenvector associated with the eigenvalue $\lambda_1 = -1$. We have

$$A + E = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}.$$

So the system $(A + E)\vec{e} = 0$ equivalent to the equation $e_1 + e_2 = 0$. Hence our eigenvector is $e = (1, -1)$. Therefore the solution associated with this eigenvalue is $x_1(t) = e^{-t}(1, -1)$. Lets find an eigenvector associated with the eigenvalue $\lambda_2 = 4$. We have

$$A - 3E = \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}.$$

So the system $(A - 4E)\vec{e} = 0$ equivalent to the equation $2e_1 - 3e_2 = 0$. Hence our eigenvector is $e = (3, 2)$. Therefore the solution associated with this eigenvalue is $x_1(t) = e^{4t}(3, 2)$. Hence the fundamental matrix

$$\Phi(t) = \begin{pmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{pmatrix}.$$

Next we find the matrix $\Phi^{-1}(t)$. Note that $det\Phi(t) = 5e^{3t}$. Then

$$\Phi^{-1}(t) = \frac{1}{5e^{3t}} \begin{pmatrix} 2e^{4t} & -3e^{4t} \\ e^{-t} & e^{-t} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2e^{t} & -3e^{t} \\ e^{-4t} & e^{-4t} \end{pmatrix}.$$

Then we have

$$\Phi^{-1}(t)f(t) = \frac{1}{5}((10+2t)e^t, (5-2t)e^{-4t}), \quad \int \Phi^{-1}(t)f(t)dt = \frac{1}{5}((2t-8)e^t, (\frac{t}{2} - \frac{9}{8})e^{-4t}).$$

Finally

$$x_p(t) = \Phi(t) \int \Phi^{-1}(t)f(t)dt = \frac{1}{5}((2t-8) + 3(\frac{t}{2} - \frac{9}{8}), -(2t-8) + 2(\frac{t}{2} - \frac{9}{8})).$$

The general solution is

$$X(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} (2t-8) + 3(\frac{t}{2} - \frac{9}{8}) \\ -(2t-8) + 2(\frac{t}{2} - \frac{9}{8}) \end{pmatrix}.$$