LECTURE 41 (267)

Homogeneous systems of linear ordinary differential equations. Defective eigenvalues.

Today continue to study the defective eigenvalues of the matrix A. First we consider the following situation: Let λ be an eigenvalue of the multiplicity $k = 3$. With this eigenvalue associated two linearly independent eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Therefore $p = 2$ and our eigenvalue is defective. Then the set of linearly independent solution associated with this eigenvalue given by the following formulae:

$$x_1(t) = e^{\lambda t}\vec{v}_1, \quad x_2(t) = e^{\lambda t}\vec{v}_2, \quad x_3(t) = ((c_1\vec{v}_1 + c_2\vec{v}_2)t + \vec{v}_3)e^{\lambda t},$$

where vector $\vec{v}_3$ solves the following system

$$(A - \lambda E)\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2.$$

Here c_1, c_2 are unknown constants and $\vec{v}_3$ is unknown vector.

Example 1. Find a general solution to the system of ordinary differential equations

$$\frac{dX}{dt} = AX, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ -2 & -4 & -1 \end{pmatrix}.$$

Solution. We have

$$A - \lambda E = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix}$$

Then

$$det(A - \lambda E) = det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix} = -(\lambda - 1)^3.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Therefore the matrix A has only one eigenvalue $\lambda = 1$. Lets find eigenvectors of the matrix A We have

$$A - E = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix}$$

Therefore the system $(A-E)\vec{e} = 0$ Is equivalent to the equation $e_1+2e_2+e_3 = 0$. Then we have two linearly independent eigenvectors $\vec{v}_1 = (1, 0, -1)$ and $\vec{v}_2 = (2, -1, 0)$. Consider the system

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix} \vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2.$$

Therefore we have the system

$$\begin{cases} 0 = c_1 + 2c_2 \\ a + 2b + c = -c_2 \\ -2a - 4b - 2c = -c_1. \end{cases}$$

Then

$$\begin{cases} c_1 = -2c_2 \\ a + 2b + c = -c_2 \\ -2a - 4b - 2c = 2c_2. \end{cases}$$

Therefore

$$\begin{cases} c_1 = -2c_2 \\ a + 2b + c = -c_2. \end{cases}$$

We set $c_2 = 1, c_1 = -2, a = -1, b = c = 0$. Then the general solution is

$$X(t) = C_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} e^t + C_3 \begin{pmatrix} -1 \\ -t \\ 2t \end{pmatrix} e^t.$$

Now we consider the situation when our matrix A has an eigenvalue λ of multiplicity $k = 3$ and only one linearly independent eigenvector. Therefore $p = 1$. In this situation there exists a vector $\vec{v}_3$ such that

$$(A - \lambda E)^3\vec{v}_3 = 0, \quad (A - \lambda E)^2\vec{v}_3 = \vec{v}_1 \neq 0, \quad (A - \lambda E)\vec{v}_3 = \vec{v}_2 \neq 0.$$

The set of linearly independent solutions associated with the eigenvalue λ is

$$x_1(t) = e^{\lambda t}\vec{v}_1, \quad x_2(t) = e^{\lambda t}(\vec{v}_1 t + \vec{v}_2), \quad x_3(t) = (\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3)e^{\lambda t},$$

Example 2. Find a general solution to the system of linear ordinary differential equations $\frac{dX}{dt} = AX$ where

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{pmatrix}$$

Solution. We have

$$A - \lambda E = \begin{pmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -4 \\ 0 & 1 & -3-\lambda \end{pmatrix}$$

Then

$$det(A - \lambda E) = det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 3-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix} = -(\lambda+1)^3.$$

Therefore the matrix A has only one eigenvalue $\lambda = -1$. Lets find eigenvectors of the matrix A We have

$$A + E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & -1 & -2 \end{pmatrix}$$

Therefore the system $(A+E)\vec{e} = 0$ Is equivalent to the system $e_3 = 0, \quad 2e_2 - 4e_3 = 0, \quad -e_2 - 2e_3 = 0$. Then we have $e_2 = e_3 = 0$ and the matrix A has only one eigenvector $\vec{e} = (1, 0, 0)$. Therefore in our situation $k = 3$ and $p = 1$. So we need to find a vector $\vec{v}_3$. We have

$$(A+E)^2 = (A+E)(A+E) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Next we find the matrix

$$(A+E)^3 = (A+E)\cdot(A+E)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

As a vector $\vec{v}_3$ we take $\vec{v}_3 = (0,0,1)$. In that case $\vec{v}_2 = (A+E)\vec{v}_3 = (1,-4,-2), \vec{v}_1 = (-2,0,0)$. Therefore the general solution is

$$X(t) = C_1 e^{-t} \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} e^t + C_2 \begin{pmatrix} -2t+1 \\ -4 \\ -2 \end{pmatrix} e^{-t} + C_3 \begin{pmatrix} -t^2+t \\ -4t \\ -2t+1 \end{pmatrix} e^{-t}.$$