

## LECTURE 40 (267)

### Defective eigenvalues

Let  $A$  be a matrix of the size  $n \times n$ . We consider the polynomial  $\det(A - \lambda E)$ . This is the polynomial of degree  $n$ . The roots of the polynomial  $\det(A - \lambda E)$  are an eigenvalues of the matrix  $A$ . Let  $\lambda_j$  be an eigenvalue of the matrix  $A$ . By **multiplicity** of the eigenvalue  $\lambda_j$  we mean the multiplicity of the root  $\lambda_j$  of the polynomial  $\det(A - \lambda E)$ . We denote the multiplicity of the eigenvalue  $\lambda_j$  as  $k$ . Let  $p$  be the number of linearly independent eigenvectors associated with the eigenvalue  $\lambda_j$ . If  $p < k$  an eigenvalue  $\lambda_j$  called **defective**.

Today we concentrate on the following situation: an eigenvalue  $\lambda_j$  has the multiplicity  $k = 2$  and number of linearly independent eigenvectors  $p = 1$ . We denote such an eigenvector as  $v_1$ . How to construct the set of linearly independent solutions associated with the eigenvalue  $\lambda_j$ . In this situation the set of solutions given by the following formulae

$$x_1(t) = v_1 e^{\lambda_j t}, \quad x_2(t) = (v_1 t + v_2) e^{\lambda_j t}, \quad (1)$$

where vector  $v_2$  solves the system

$$(A - \lambda_j E)v_2 = v_1. \quad (2)$$

**Example 1.** Find the general solution of the system  $\frac{dx}{dt} = Ax$

$$A = \begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix}$$

*Solution.* We have

$$A - \lambda E = \begin{pmatrix} -2 - \lambda & 1 \\ -1 & -4 - \lambda \end{pmatrix}$$

Then

$$\det(A - \lambda E) = \det \begin{pmatrix} -2 - \lambda & 1 \\ -1 & -4 - \lambda \end{pmatrix} = (-2 - \lambda)(-4 - \lambda) + 1 = \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$$

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therefore the matrix  $A$  has only one eigenvalue  $\lambda = -3$ . Lets find eigenvectors of the matrix  $A$  We have

$$A + 3E = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Therefore the system  $(A+3E)\vec{v}_1 = 0$ . Is equivalent to the equation  $e_1 + e_2 = 0$ . Then vector  $\vec{v}_1 = (1, -1)$  is the eigenvector of the matrix  $A$ . So we have only one linearly independent eigenvector associated with the eigenvalue  $\lambda = -3$ . Therefore  $k = 2$  and  $p = 1$ . Hence our eigenvalue is **defective** . Lets find vector  $v_2 = (a, b)$ . We have to solve the system (2):

$$a + b = 1, \quad -a - b = -1.$$

We put  $a = 1$  and  $b = 0$ . So  $v_2 = (1, 0)$ . Hence

$$X(t) = C_1 x_1(t) + C_2 x_2(t), \quad x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}, \quad x_2(t) = \begin{pmatrix} t+1 \\ -t \end{pmatrix} e^{-3t}.$$

**Example 2.** Find the general solution of the system  $\frac{dx}{dt} = Ax$

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

*Solution.* We have

$$A - \lambda E = \begin{pmatrix} 3 - \lambda & -1 \\ 1 & 1 - \lambda \end{pmatrix}$$

Then

$$\det(A - \lambda E) = \det \begin{pmatrix} 3 - \lambda & -1 \\ 1 & 1 - \lambda \end{pmatrix} = (3 - \lambda)(1 - \lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

Therefore the matrix  $A$  has only one eigenvalue  $\lambda = 2$ . Lets find eigenvectors of the matrix  $A$ . We have

$$A - 2E = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

Therefore the system  $(A-2E)\vec{v}_1 = 0$  Is equivalent to the equation  $e_1 - e_2 = 0$ . Then vector  $\vec{v}_1 = (1, 1)$  is the eigenvector of the matrix  $A$ . So we have only one linearly independent eigenvector associated with the eigenvalue  $\lambda = 2$ .

Therefore  $k = 2$  and  $p = 1$ . Hence our eigenvalue is defective. Lets find vector  $v_2 = (a, b)$ . We have to solve the system (2):

$$a - b = 1, \quad a - b = 1.$$

We put  $a = 1$  and  $b = 0$ . So  $v_2 = (1, 0)$ . Hence

$$X(t) = C_1 x_1(t) + C_2 x_2(t), \quad x_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}, \quad x_2(t) = \begin{pmatrix} t+1 \\ t \end{pmatrix} e^{2t}.$$

**Example 3.** Find the general solution of the system  $\frac{dx}{dt} = Ax$

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$$

*Solution.* We have

$$A - \lambda E = \begin{pmatrix} 1 - \lambda & -2 \\ 2 & 5 - \lambda \end{pmatrix}$$

Then

$$\det(A - \lambda E) = \det \begin{pmatrix} 1 - \lambda & -2 \\ 2 & 5 - \lambda \end{pmatrix} = (1 - \lambda)(5 - \lambda) + 4 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

Therefore the matrix  $A$  has only one eigenvalue  $\lambda = 3$ . Lets find eigenvectors of the matrix  $A$  We have

$$A - 3E = \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}$$

Therefore the system  $(A - 3E)\vec{e} = 0$  Is equivalent to the equation  $e_1 + e_2 = 0$ . Then vector  $\vec{v}_1 = (1, -1)$  is the eigenvector of the matrix  $A$ . So we have only one linearly independent eigenvector associated with the eigenvalue  $\lambda = 3$ . Therefore  $k = 2$  and  $p = 1$ . Hence our eigenvalue is defective. Lets find vector  $v_2 = (a, b)$ . We have to solve the system (2):

$$-2a - 2b = 1, \quad 2a + 2b = 1.$$

We put  $a = -\frac{1}{2}$  and  $b = 0$ . So  $v_2 = (1, 0)$ . Hence

$$X(t) = C_1 x_1(t) + C_2 x_2(t), \quad x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}, \quad x_2(t) = \begin{pmatrix} t - \frac{1}{2} \\ -t \end{pmatrix} e^{3t}.$$